1 球函数

1.1 拉普拉斯方程

$$\Delta u = 0$$
$$u(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

$$\begin{cases} r^2R'' + 2rR' - \lambda R = 0 \ (\lambda = l(l+1)) \\ \sin^2\theta Y''_\theta + Y''_\phi + \sin\theta\cos\theta Y'_\theta + \lambda\sin^2\theta Y = 0 \end{cases}$$

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

$$\begin{cases} r^2R'' + 2rR' - l(l+1)R = 0 \\ \frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + [l(l+1) - \frac{m^2}{\sin^2\theta}]\Theta = 0 \ \textbf{Legendre Equation} \\ \Phi'' + m^2\Phi = 0 \end{cases}$$

Legendre Equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + [l(l+1) - \frac{m^2}{\sin^2\theta}]\Theta = 0$$
$$x = \cos\theta;\ \sin^2\theta = 1 - \cos^2\theta = 1 - x^2;\ dx = -\sin\theta d\theta$$
$$\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}] + [l(l+1) - \frac{m^2}{1-x^2}]\Theta = 0 \text{ 连带勒让德方程}$$
$$\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}] + [l(l+1)]\Theta = 0 \text{ 勒让德方程}$$

$$m = 0$$
$$\Theta''(x) - \frac{2x}{1-x^2}\Theta'(x) + \frac{l(l+1)}{1-x^2}\Theta(x) = 0$$
$$\Theta(x) = P_l(x) = P_l(\cos\theta)$$

$$m \neq 0$$
$$\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}] + [l(l+1) - \frac{m^2}{1-x^2}]\Theta = 0$$
$$\Theta(x) = y(x)(1-x^2)^{m/2} = P_l^{[m]}(x)(1-x^2)^{m/2} = P_l^m(x) = P_l^m(\cos\theta)$$

$$\begin{cases} R(r) = \begin{Bmatrix} r^l \\ 1/r^{l+1} \end{Bmatrix} \\ \Theta(\cos\theta) = P_l^m(\cos\theta) \\ \Phi(\phi) = \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \end{cases}$$

$$u(r,\theta,\phi) = \sum_{l,m} R_l(r)\Phi_m(\phi)P_l^m(\cos\theta)$$

1.2 泊松方程

$$\Delta u = F(r,\theta,\phi)$$
$$u(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

$$\begin{cases} \frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + [l(l+1) - \frac{m^2}{\sin^2\theta}]\Theta = 0 \\ \Phi'' + m^2\Phi = 0 \end{cases}$$

$$u(r,\theta,\phi)=\sum_{l,m}R_{l,m}(r)Y_{l,m}(\theta,\phi)$$

$$F(r,\theta,\phi)=\sum_{l,m}f_{l,m}(r)Y_{l,m}(\theta,\phi)$$

$$\Delta R_{l,m}(r)=f_{l,m}(r)$$

$$\begin{cases} R(r) \\ \Theta(\cos\theta)=P_l^m(\cos\theta) \\ \Phi(\phi)=\begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \end{cases}$$

$$u(r,\theta,\phi)=\sum_{l,m}R_{l,m}(r)Y_{l,m}(\theta,\phi)$$

1.3 亥姆霍兹方程

$$\Delta v+k^2v=0$$

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr})+[k^2-l(l+1)]R=0 \quad \textbf{球 Bessel Equation} \\ \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial Y}{\partial\phi}+l(l+1)Y=0 \end{cases}$$

球 Bessel Equation

$$k\neq 0$$

$$x=kr;\quad R(x)=\sqrt{\frac{\pi}{2x}}y(x)$$

$$y(x)=\begin{Bmatrix} \begin{cases} J_{l+1/2}(x) \\ J_{-(l+1/2)}(x) \end{cases} \\ N_{l+1/2}(x) \end{Bmatrix} or \begin{Bmatrix} H_{l+1/2}^{(1)}(x) \\ H_{l+1/2}^{(2)}(x) \end{Bmatrix}$$

$$R(r)=\begin{Bmatrix} j_l(kr) \\ n_l(kr) \end{Bmatrix} or \begin{Bmatrix} h_l^{(1)}(kr) \\ h_l^{(2)}(kr) \end{Bmatrix}$$

$$k=0$$

$$R(r)=\begin{Bmatrix} r^l \\ 1/r^{l+1} \end{Bmatrix}$$

$$\begin{cases} R(r) \\ \Theta(\cos\theta)=P_l^m(\cos\theta) \\ \Phi(\phi)=\begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \end{cases}$$

$$v(r,\theta,\phi)=\sum_{k,l,m}R_{k,l}(r)\Phi_m(\phi)P_l^m(\cos\theta)$$

---

2 柱函数

2.1 拉普拉斯方程

$$\Delta u = 0$$
$$u(\rho,\phi,z) = R(\rho)\Phi(\phi)Z(z)$$

$$\begin{cases} \rho^2 R'' + \rho R' + [\mu\rho^2 - m^2]R = 0 \begin{cases} \mu > 0, & \textbf{Beseel Equation} \\ \mu < 0, & \textbf{虚宗量Beseel Equation} \end{cases} \\ \Phi'' + m^2\Phi = 0 \\ Z'' - \mu Z = 0 \end{cases}$$

Beseel Equation

$$\mu = 0$$

$$\rho^2 R'' + \rho R' - m^2 R = 0$$

$$R_m(\rho) = \begin{Bmatrix} \rho^m \\ 1/\rho^m \end{Bmatrix}, \ R_0(\rho) = \begin{Bmatrix} 1 \\ \ln\rho \end{Bmatrix}$$

$$Z(z) = \begin{Bmatrix} 1 \\ z \end{Bmatrix}$$

$$u(\rho,\phi,z) = \sum_m R_m(\rho)\Phi_m(\phi)Z(z)$$

$$\mu > 0$$

$$\rho^2 R'' + \rho R' + [(\sqrt{\mu}\rho)^2 - m^2]R = 0$$

$$x = \sqrt{\mu}\rho$$

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - m^2)R = 0$$

$$R(\rho) = \begin{Bmatrix} J_m(\sqrt{\mu}\rho) \\ N_m(\sqrt{\mu}\rho) \end{Bmatrix} or \begin{Bmatrix} H_m^{(1)}(\sqrt{\mu}\rho) \\ H_m^{(2)}(\sqrt{\mu}\rho) \end{Bmatrix}$$

$$Z(z) = \begin{Bmatrix} e^{\sqrt{\mu}\rho} \\ e^{-\sqrt{\mu}\rho} \end{Bmatrix}$$

$$u(\rho,\phi,z) = \sum_{m,\mu} R_{m,\mu}(\rho)\Phi_m(\phi)Z_\mu(z)$$

$$\mu < 0$$

$$\rho^2 R'' + \rho R' + [(\sqrt{-\mu}\rho)^2 - m^2]R = 0$$

$$x = \sqrt{-\mu}\rho = h\rho$$

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}-(x^2+m^2)R=0$$

$$R(\rho)=\left\{\begin{array}{c} I_m(h\rho) \\ \\ K_m(h\rho) \end{array}\right\}$$

$$Z(z)=\left\{\begin{array}{c} \cos hz \\ \\ \sin hz \end{array}\right\}$$

$$u(\rho,\phi,z)=\sum_{m,h}R_{m,h}(\rho)\Phi_m(\phi)Z_h(z)$$

$$\begin{cases} R(\rho) \\ \\ Z(Z) \\ \\ \Phi(\phi)=\left\{\begin{array}{c} \cos m\phi \\ \\ \sin m\phi \end{array}\right\} \end{cases}$$

2.2 泊松方程

2.3 亥姆霍兹方程

$$\Delta v+k^2v=0$$

$$-\mu=h^2$$

$$\begin{cases} \rho^2R''+\rho R'+[(k^2-h^2)\rho^2-m^2]R=0\begin{cases} k^2-h^2>0, & \textbf{Beseel Equation} \\ \\ k^2-h^2<0, & \textbf{虚宗量Beseel Equation} \end{cases} \\ \\ \Phi''+m^2\Phi=0 \\ \\ Z''-\mu Z=0 \end{cases}$$

$$h=0$$

$$Z(z)=\left\{\begin{array}{c} 1 \\ \\ z \end{array}\right\}$$

$$k=h=0$$

$$R_0(\rho)=\left\{\begin{array}{c} 1 \\ \\ \ln\rho \end{array}\right\};\ R_m(\rho)=\left\{\begin{array}{c} \rho^m \\ \\ \rho^{-m} \end{array}\right\}$$

$$h\neq 0$$

$$\begin{cases} R(\rho)=\left\{\begin{array}{c} J_m(\sqrt{k^2-h^2}\rho) \\ \\ N_m(\sqrt{k^2-h^2}\rho) \end{array}\right\} or \left\{\begin{array}{c} H_m^{(1)}(\sqrt{k^2-h^2}\rho) \\ \\ H_m^{(2)}(\sqrt{k^2-h^2}\rho) \end{array}\right\} \\ \\ \Phi(\phi)=\left\{\begin{array}{c} \cos m\phi \\ \\ \sin m\phi \end{array}\right\} \\ \\ Z(z)=\left\{\begin{array}{c} \cos hz \\ \\ \sin hz \end{array}\right\} \end{cases}$$

$$u(\rho,\phi,z)=\sum_{k,m,h}R_{k,m,h}(\rho)\Phi_m(\phi)Z_h(z)$$

---

3 本征函数

3.1 勒让德多项式/勒让德函数

$$\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}]+l(l+1)\Theta=0$$

由级数法推出

$$\Theta(x)=a_0\Theta_1(x)+a_1\Theta_2(x)$$

当l为整数时得到独立解-第一类勒让德函数

$$P_l(x)=\sum_{k=0}^{[l/2]}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k}$$

另一线性独立解为-第二类勒让德函数

$$Q_l(x)=P_l(x)\int\frac{1}{(1-x^2)[P_l(x)]^2}dx$$

勒让德函数的其他表示

- 微分表示

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

- 积分表示

$$P_l(x)=\frac{1}{2\pi i}\frac{1}{2^l}\oint_C\frac{(z^2-1)^l}{(z-x)^{l+1}}dz$$

本征函数性质

- 正交性
- 模

$$N_l^2=\frac{2}{2l+1}$$

- 广义傅里叶级数

$$f(x)=\sum_{l=0}^{\infty}f_lP_l(x)$$

$$f_l=\frac{2l+1}{2}\int_{-1}^{1}f(x)P_l(x)dx \tag{1}$$

$$f(\theta)=\sum_{l=0}^{\infty}f_lP_l(\cos\theta)$$

$$f_l=\frac{2l+1}{2}\int_{0}^{\pi}f(\theta)P_l(\cos\theta)\sin\theta d\theta \tag{2}$$

母函数

$$\frac{1}{\sqrt{R^2-2rR\cos\theta+r^2}}=\begin{cases}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta) & (r<R),\\ \sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}P_l(\cos\theta) & (r>R).\end{cases}$$

递推公式

$$(k+1)P_{k+1}(x)-(2k+1)xP_k(x)+kP_{k-1}(x)=0\ \ (k\geq 1)$$

$$P_k(x)=P'_{k+1}(x)-2xP'_k(x)+P'_{k-1}(x)\ \ (k\geq 1)$$

$$(k+1)P'_{k+1}(x)-(2k+1)P_k(x)-(2k+1)xP'_k(x)+kP'_{k-1}(x)=0\ \ (k\geq 1)$$

$$(2k+1)P_k(x)=P'_{k+1}(x)-P'_{k-1}(x)\ \ (k\geq 1)$$

$$P'_{k+1}(x)=(k+1)P_k(x)+xP'_k(x)$$

$$kP_k(x)=xP'_k(x)-P'_{k-1}(x)\ \ (k\geq 1)$$

$$(x^2-1)P'_k(x)=kxP_k(x)-kP_{k-1}(x)\ \ (k\geq 1)$$

常用值

$$\begin{aligned}&P_0(x)=1\\&P_1(x)=x\\&P_2(x)=\tfrac{1}{2}(3xP_1(x)-P_0(x))=\tfrac{1}{2}\left(3x^2-1\right)\\&P_3(x)=\tfrac{1}{3}(5xP_2(x)-2P_1(x))=\tfrac{1}{2}\left(5x^3-3x\right)\\&P_4(x)=\tfrac{1}{4}(7xP_3(x)-3P_2(x))=\tfrac{1}{8}\left(35x^4-30x^2+3\right)\\&P_5(x)=\tfrac{1}{5}(9xP_4(x)-4P_3(x))=\tfrac{1}{8}\left(63x^5-70x^3+15x\right)\end{aligned}$$

$$P_{2n}(0)=\frac{(-1)^n}{2^{n+1}}(2^n-1)$$

$$P_l(1)=1;\ \ P_l(-1)=(-1)^l$$

3.2 连带勒让德函数

$$\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}]+[l(l+1)-\frac{m^2}{1-x^2}]\Theta=0$$

解得

$$\Theta(x)=P_l^{[m]}(1-x^2)^{m/2}=\frac{1}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l=P_l^m(x)$$

线性非独立

$$P_l^m(x)=CP_l^{-m}(x)$$

$$C=(-1)^m\frac{(l+m)!}{(l-m)!}$$

本征函数性质

- 正交性
- 模

$$(N_l^m)^2 = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$$

- 广义傅里叶级数展开

$$f_l = \frac{1}{(N_l^m)^2}\int_{-1}^{1} F(x)P_l^m(x)dx$$

递推公式

$$(2k+1)xP_k^m(x) = (k+m)P_{k-1}^m(x) + (k-m+1)P_{k+1}^m(x) \;\; (k \geq 1)$$

$$(2k+1)(1-x^2)^{1/2}P_k^m(x) = P_{k+1}^{m+1} - P_{k-1}^{m+1} \;\; (k \geq 1)$$

$$(2k+1)(1-x^2)^{1/2}P_k^m(x) = (k+m)(k+m-1)P_{k-1}^{m-1} - (k-m+2)(k-m+1)P_{k+1}^{m+1} \;\; (k \geq 1)$$

$$(2k+1)(1-x^2)\frac{dP_k^m(x)}{dx} = (k+1)(k+m)P_{k-1}^m - k(k-m+1)P_{k+1}^m \;\; (k \geq 1)$$

常用值

- **端点值**：对任意 l, m 有
  $P_l^m(1) = \delta_{m,0} \qquad P_l^m(-1) = (-1)^l\delta_{m,0}$

3.3 球谐函数

- **三角形式**

$$Y_{l,m}(\theta,\phi) = P_l^m(\cos\theta)\begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$$

- **复形式**

$$Y_{l,m}(\theta,\phi) = P_l^{|m|}(\cos\theta)e^{im\phi}$$

本征函数性质

- 正交性
- 模
  - 三角形式

$$(N_l^m)^2 = \frac{2\pi\delta_m}{2l+1}\frac{(l+m)!}{(l-m)!}, \begin{cases} 2 & (m=0) \\ 1 & (m=1,2,3,\dots) \end{cases}$$

  - 复形式

$$(N_l^m)^2 = \frac{4\pi}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}$$

- 广义傅里叶级数展开

加法公式

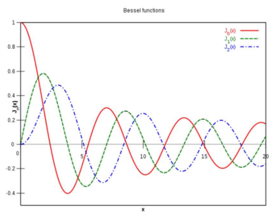
$$P_l(\cos\Theta)=\sum_{m=-l}^{+l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_0)P_l^m(\cos\theta)e^{im(\phi-\phi_0)}$$

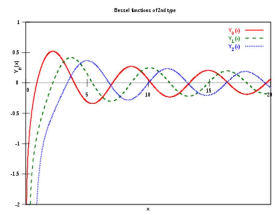
3.4 贝塞尔函数

$$J_m(x)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(m+k+1)}\frac{x}{2}^{m+2k}$$

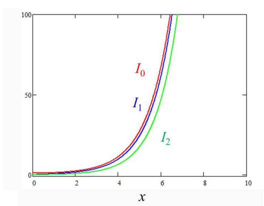
$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

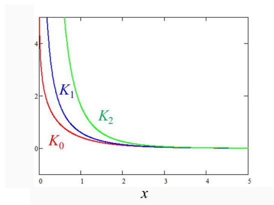
$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$





$$I_m(x)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(m+k+1)}\frac{x}{2}^{m+2k}$$





$$j_\ell(x)\equiv(-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\frac{\sin x}{x}$$

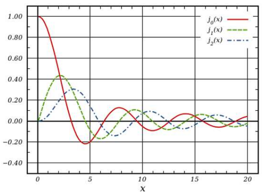
$$j_0(x)=\frac{\sin(x)}{x}$$

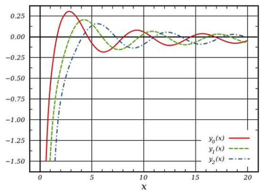
$$j_1(x)=\frac{\sin(x)}{x^2}-\frac{\cos(x)}{x}$$

$$j_2(x)=\left(\frac{3}{x^2}-1\right)\frac{\sin(x)}{x}-\frac{3\cos(x)}{x^2}$$

$$y_\ell(x) \equiv -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos x}{x}$$

$$y_0(x) = -j_{-1}(x) = -\frac{\cos(x)}{x}$$
$$y_1(x) = j_{-2}(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x}$$
$$y_2(x) = -j_{-3}(x) = \left(-\frac{3}{x^2}+1\right)\frac{\cos(x)}{x} - \frac{3\sin(x)}{x^2}$$





本征函数性质

- 正交性
- 模

$$(N_n^{(m)})^2 = \int_0^\rho J_m^2(\frac{x_n^{(m)}}{\rho_0}\rho)\rho d\rho$$

$$= \frac{1}{2\mu_n^{(m)}}\int_0^{x_0} J_m^2(x)dx^2$$

$$= \frac{1}{2\mu_n^{(m)}}[(x^2-m^2)J_m^2(x) + x^2(J_m'(x))^2]\Big|_0^{x_0}$$

- 广义傅里叶级数展开

$$f_k = \frac{1}{(N_n^{(m)})^2}\int_0^\rho F(\rho)J_m(\frac{x_n^{(m)}}{\rho_0}\rho)\rho d\rho$$

母函数

$$e^{\frac{x}{2}(z-1/z)} = \sum_{-\infty}^{\infty} J_m(x)z^m = \sum_{-\infty}^{\infty} J_m(x)e^{im\phi}$$

加法公式

$$J_m(a+b) = \sum_{-\infty}^{\infty} J_k(a)J_{m-k}(b)$$

递推关系

$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x)$$

4　泊松方程&格林函数

- 第一格林公式

$$\oiint u\nabla v dS = \iiint_T \nabla u \nabla v dV + \iiint_T u\nabla^2 v dV$$

$$\oiint v\nabla u dS = \iiint_T \nabla u \nabla v dV + \iiint_T v\nabla^2 u dV$$

- 第二格林公式

$$\oiint (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS = \iiint_T (u\nabla^2 v - v\nabla^2 u)dV$$

$$\begin{cases} \Delta u(\mathbf{r}) = f(\mathbf{r}) \\ (\alpha u + \beta\frac{\partial u}{\partial n})\Big|_\Sigma = \phi(M) \end{cases} \quad \begin{cases} \Delta v(\mathbf{r},\mathbf{r_0}) = \delta(\mathbf{r}-\mathbf{r_0}) \\ (\alpha v + \beta\frac{\partial v}{\partial n})\Big|_\Sigma = 0 \end{cases}$$

第一类边界条件

$$\begin{cases} \Delta u(\mathbf{r}) = f(\mathbf{r}) \\ \alpha u\Big|_\Sigma = \phi(M) \end{cases} \quad \begin{cases} \Delta v(\mathbf{r},\mathbf{r_0}) = \delta(\mathbf{r}-\mathbf{r_0}) \\ \alpha v\Big|_\Sigma = 0 \end{cases}$$

$$u(\mathbf{r}) = \iiint_T v(\mathbf{r},\mathbf{r_0}) f(\mathbf{r_0}) d^3\mathbf{r_0} + \frac{1}{\alpha}\oiint_\Sigma \phi(M_0)\frac{\partial v(\mathbf{r},\mathbf{r_0})}{\partial n} d^2\mathbf{r_0}$$

第三类边界条件

$$\begin{cases} \Delta u(\mathbf{r}) = f(\mathbf{r}) \\ (\alpha u + \beta\frac{\partial u}{\partial n})\Big|_\Sigma = \phi(M) \end{cases} \quad \begin{cases} \Delta v(\mathbf{r},\mathbf{r_0}) = \delta(\mathbf{r}-\mathbf{r_0}) \\ (\alpha v + \beta\frac{\partial v}{\partial n})\Big|_\Sigma = 0 \end{cases}$$

$$u(\mathbf{r}) = \iiint_T v(\mathbf{r},\mathbf{r_0}) f(\mathbf{r_0}) d^3\mathbf{r_0} - \frac{1}{\beta}\oiint_\Sigma \phi(M_0)\frac{\partial v(\mathbf{r},\mathbf{r_0})}{\partial n} d^2\mathbf{r_0}$$

第二类边界条件

$$\begin{cases} \Delta u(\mathbf{r}) = f(\mathbf{r}) \\ \beta\frac{\partial u}{\partial n}\Big|_\Sigma = \phi(M) \end{cases} \quad \begin{cases} \Delta v(\mathbf{r},\mathbf{r_0}) = \delta(\mathbf{r}-\mathbf{r_0}) - \frac{1}{V_T} \\ \beta\frac{\partial v}{\partial n}\Big|_\Sigma = 0 \end{cases}$$

$$u(\mathbf{r}) = \iiint_T v(\mathbf{r},\mathbf{r_0}) f(\mathbf{r_0}) d^3\mathbf{r_0} - \frac{1}{\beta}\oiint_\Sigma \phi(M_0)\frac{\partial v(\mathbf{r},\mathbf{r_0})}{\partial n} d^2\mathbf{r_0} - \bar{u}$$

4.1 格林函数求解方法

无界系统

- 三维

$$G_0 = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r_0}|}$$

- 二维

$$G_0 = -\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r_0}|$$

电像法

$$\begin{cases} \Delta G(\mathbf{r},\mathbf{r_0}) = \delta(\mathbf{r}-\mathbf{r_0}) - \dfrac{1}{V_T} \\ G\Big|_{\Sigma} = 0 \end{cases}$$

$$G = G_0 + G_1$$

$$\begin{cases} \Delta G_0(\mathbf{r},\mathbf{r_0}) = \delta(\mathbf{r}-\mathbf{r_0}) - \dfrac{1}{V_T}, \quad \Delta G_1(\mathbf{r},\mathbf{r_0}) = 0 \\ G_0\Big|_{\Sigma} = -G_1\Big|_{\Sigma} \end{cases}$$

4.2 含时格林函数

4.3 冲量定理法

---

5 积分变换

5.1 傅里叶变换

无界波动（初始条件已知）

- 一维：达朗贝尔公式

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, x \in \mathbf{R}, t > 0 \\ u\big|_{t=0} = \varphi(x), \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), x \in \mathbf{R} \end{cases}$$

求出的定解

$$u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,\mathrm{d}\xi$$

- 二维：二维泊松公式（三维降维法）

定解问题：二维波动方程+初始条件

$$\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy}),(x,y)\in\mathbf{R}^2,t>0 \\ u|_{t=0}=\varphi(x,y),u_t|_{t=0}=\psi(x,y),(x,y)\in\mathbf{R}^2 \end{cases}$$

定解

二维齐次泊松公式：

$$u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{C_{at}^M}\frac{\varphi(X,Y)}{\sqrt{(at)^2-(X-x)^2-(Y-y)^2}}\mathrm{d}X\,\mathrm{d}Y+\frac{1}{2\pi a}\iint\limits_{C_{at}^M}\frac{\psi(X,Y)}{\sqrt{(at)^2-(X-x)^2-(Y-y)^2}}\mathrm{d}X\,\mathrm{d}Y$$

该解称为二维齐次波动问题的泊松公式

注：由三维推出二维定解问题的方法称为降维法

- 三维：三维泊松公式

定解问题： 三维波动方程+初始条件

$$\begin{cases} u_{tt}=a^2\Delta u+f(M,t), \quad M(x,y,z)\in\mathbf{R}^3,t>0 \\ u|_{t=0}=\varphi(M),u_t|_{t=0}=\psi(M),M\in\mathbf{R}^3 \end{cases}$$

定解

三维齐次泊松公式：

$$u(M,t)=\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\iint\limits_{S_{at}^M}\frac{\varphi(X,Y,Z)}{t}\mathrm{d}S+\frac{1}{4\pi a^2}\iint\limits_{S_{at}^M}\frac{\psi(X,Y,Z)}{t}\mathrm{d}S$$

其中

$$S_{at}^M=\{(X,Y,Z)\mid(X-x)^2+(Y-y)^2+(Z-z)^2=(at)^2\}$$

无界波动（受迫）

- 推迟势

$$u(\mathbf{r},t)=\frac{1}{4\pi a^2}\iiint_{T_{at}^r}\frac{[f]}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{V}'$$

$$[f]=f(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/a)$$

定解问题：三维非齐次波动方程 + 初始条件

$$\begin{cases} u_{tt}=a^2\Delta u+f(M,t), \quad M(x,y,z)\in \mathbf{R}^3, t>0 \\ u|_{t=0}=\varphi(M), u_t|_{t=0}=\psi(M), M\in \mathbf{R}^3 \end{cases}$$

定解，由杜哈梅原理和三维波动泊松公式求解

三维非齐次 Kirchhoff 公式：

$$u(M,t)=\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\iint_{S_{at}^M}\frac{\varphi(X,Y,Z)}{t}\mathrm{d}S+\frac{1}{4\pi a^2}\iint_{S_{at}^M}\frac{\psi(X,Y,Z)}{t}\mathrm{d}S$$
$$+\frac{1}{4\pi a^2}\iiint_{T_{at}^M}\frac{f\left(X,Y,Z,t-\frac{r}{a}\right)}{r}\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$$

该式称为三维非齐次波动问题的 Kirchhoff 公式。该公式的第三项由外力（或称为“源”）引起，其中 $t-\frac{r}{a}$ 表明，M 点处受到外力影响的时刻 t，比外力发出的时刻晚了 $\frac{r}{a}$。

无限扩散

- 常见积分公式

$$\int_{-\infty}^{\infty} e^{-\alpha^2k^2}e^{\beta k}dk=(\sqrt{\pi}/\alpha)e^{\beta^2/4\alpha^2}$$

半无限扩散

- 偶延拓（限定源扩散）

高斯函数

$$\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4a^2t}}$$

- 奇延拓（恒浓度扩散）

误差函数

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$$

余误差函数

$$erfc(x)=1-erf(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-z^2}dz$$

5.2 拉普拉斯变换

---

6 保角变换

$$\begin{cases} \Delta u(\mathbf{r}) = f(\mathbf{r}) \\ (\alpha u + \beta \frac{\partial u}{\partial n})\Big|_{\Sigma} = \phi(M) \end{cases}$$

$$(x, y) \longrightarrow (\xi, \eta)$$

$$z \longrightarrow \zeta(z) = \xi(x, y) + i\eta(x, y)$$

$$\Delta u(\xi, \eta) = f(\bar{\xi}, \eta) = \frac{1}{[\zeta'(z)]^2} f(x(\xi, \eta), y(\xi, \eta)) \quad (\zeta'(z) \neq 0)$$

线性变换

$$\zeta(z) = az + b = |a|e^{iarga}(z + b/a) = |a|e^{iarga}z_1 = |a|z_2$$

幂函数、根式

- $$\zeta(z) = z^n$$

  $$原点处：arg\zeta = argz^n = nargz$$

  $$\zeta(z) = \sqrt[n]{z}$$

  $$原点处：arg\zeta = argz^{\frac{1}{n}} = \frac{1}{n}argz$$

指数、对数

- $$\zeta(z) = e^z = e^x e^{iy} = |\zeta|e^{iarg\zeta}$$

  $$z = |z|e^{iargz}$$

- $$\zeta(z) = \ln z = \ln|z| + iargz$$

  $$z = x + iy$$

反演变换

$$\zeta(z) = \frac{R^2}{z} = \frac{R^2}{\rho}e^{-i\phi} = [\frac{R^2}{\rho}e^{i\phi}]* = z_1*$$

$$|z||z_1*| = R^2$$

共形变换（分式线性）

$$\zeta(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)$$
$$= \frac{a}{c} + \frac{(bc - ad)/c^2}{z + d/c} = \frac{a}{c} + \frac{(bc - ad)/c^2}{z_1} = \frac{a}{c} + z_2$$

儒阔夫斯基变换

$$\zeta(z) = \frac{1}{2}(z + \frac{1}{z})$$

$$同心圆族：|z| = \rho_0$$

$$同心椭圆族：\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

$$a = \frac{1}{2}(\rho_0 + \frac{1}{\rho_0}), \;\; b = \frac{1}{2}(\rho_0 - \frac{1}{\rho_0})$$