数理方法.

§1.1 复数及运算.

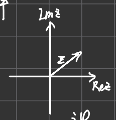
1. 起源.

$$x^3+ax^2+bx+c=0$$
$$\Big\Downarrow\ x=t-\frac{a}{3}$$
$$t^3+pt+q=0$$
$$\Big\Downarrow\ t=u+v$$
$$u^3+v^3+(u+v)(3uv+p)+q=0$$
$$\Big\Downarrow\ \text{assume } 3uv+p=0$$
$$\begin{cases} u^3+v^3+q=0 \\ uv=-\frac{p}{3}\end{cases} \Longrightarrow u^6+u^3v^3+qu^3=0 \Rightarrow u^6+qu^3-\frac{1}{27}p^3=0$$
$$\Downarrow$$
$$\begin{cases} u=\sqrt[3]{\dfrac{-q\pm\sqrt{q^2+\frac{4}{27}p^3}}{2}} \\ v=-\dfrac{p}{3}\sqrt[3]{\dfrac{2}{-q\pm\sqrt{q^2+\frac{4}{27}p^3}}}\end{cases} \Longrightarrow x=u+v-\frac{a}{3}$$

2. 定义. 闭群

① $z=x+iy$

② 矢量表达



③ 极坐标 $z=\rho e^{i\varphi}$ $\arg z=\varphi+g^{-1}\left(\frac{y}{x}\right)$

④ 测地投影

3. 表示.

4. 复共轭

5. 运算规则

$z=x+iy$

$x=\mathrm{Re}\,z$

$y=\mathrm{Im}\,z$

$z=\rho e^{i\varphi}$

eg: 求根

① $\sqrt{a+ib}=C+iD$

② $z=a+ib=\rho e^{i\varphi}$

$$=\sqrt{a^2+b^2}\,e^{i\,tg^{-1}\frac{b}{a}}$$

$$\sqrt{z}=(a^2+b^2)^{\frac{1}{4}}e^{\frac{1}{2}i\,tg^{-1}\frac{b}{a}}$$

$$=(a^2+b^2)^{\frac{1}{4}}\left[\cos\tfrac{1}{2}\left(tg^{-1}\tfrac{b}{a}\right)+i\sin\tfrac{1}{2}\left(tg^{-1}\tfrac{b}{a}\right)\right]$$

$x\in\mathbb{R}$, $|\cos x|\le 1$

$z\in\mathbb{C}$, $|\cos z|$ 可能 >1

$$\cos i=\frac{e^{-1}+e^{1}}{2}>1$$
$$\Downarrow$$
$$\begin{cases}\cos(ia)=\dfrac{e^{-a}+e^{a}}{2}=\mathrm{ch}\,a \\ \sin(ib)=\dfrac{e^{-b}-e^{b}}{2i}=\dfrac{e^{b}-e^{-b}}{2}i=i\,\mathrm{sh}\,b\end{cases}$$
$$\begin{cases}\mathrm{ch}\,ia=\cos a \\ \mathrm{sh}\,ib=\sin b\end{cases}$$

§1.2 复变函数

$$x\to z=x+iy$$
$$f(z)=u(x,y)+iv(x,y)$$

Def: 如果复数z在复平面 or 复球面上的某一区域B上连续变动，若复变数w的值随着z值而变化，则称 $w=f(z)$ 为B上的复变数z的函数. z称为宗量. $w=f(z)$, $z\in B$

1) 邻域 $D_\varepsilon(z_0)=\{z \mid |z-z_0|<\varepsilon\}$ z_0的邻域

2) 区域 ①一定有邻域的内点组成
② 具有连通性，即点集中任意两点都可以用折线连接.

3) 例：① 复项式：$w=f(z)=C_0+C_1z+C_2z^2+C_3z^3=u(x,y)+iv(x,y)$　C_i:复常数

② 有理式：$w=f(z)=\dfrac{a_0+a_1z+a_2z^2+\cdots+a_nz^n}{b_0+b_1z+b_2z^2+\cdots+b_mz^m}$

③ 根式：$\sqrt[n]{z-z_0}$ $\Longrightarrow$ 多值函数　$z-z_0 \xrightarrow{\text{开3根}} \begin{cases}\sqrt[3]{z-z_0}\\ \sqrt[3]{z-z_0}\,e^{i\frac{2\pi}{3}}\\ \sqrt[3]{z-z_0}\,e^{i\frac{4\pi}{3}}\end{cases}$

④ 初等函数

$$\ln z=\ln z+i2k\pi=\ln\rho+i\varphi+i2k\pi$$

$$e^z=e^{x+iy}=e^x\cdot e^{iy}=e^x\cos y+ie^x\sin y$$

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}=-\frac{[e^{i(x+iy)}-e^{-i(x+iy)}]}{2}i=\frac{1}{2}(i\cos x+\sin x)e^{y}-\frac{1}{2}(i\cos x-\sin x)e^{-y}=\sin x\,\mathrm{ch}y+i\cos x\,\mathrm{sh}y \sim \sin(x+y)$$

$$\cos z=\frac{e^{iz}+e^{-iz}}{2}=\frac{1}{2}(\cos x+i\sin x)e^{-y}+\frac{1}{2}(\cos x-i\sin x)e^{y}=\cos x\,\mathrm{ch}y-i\sin x\,\mathrm{sh}y \sim \cos(x+y)$$

e^z, $\mathrm{sh}z$, $\mathrm{ch}z$ 均有复周期.

§1.3 复导数

1) Def：$\dfrac{df(z)}{dz}=\lim\limits_{\Delta z\to0}\dfrac{f(z+\Delta z)-f(z)}{\Delta z}$

等价于　$i\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}$
or
$\dfrac{\partial f}{\partial z^*}=0$

2) 必要条件：
柯西黎曼条件

$$f'(z)=\lim_{\Delta z\to0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\substack{\Delta x\to0\\ \Delta y=0}}\frac{f(z+\Delta x)-f(z)}{\Delta x}=\lim_{\substack{\Delta y\to0\\ \Delta x=0}}\frac{f(z+i\Delta y)-f(z)}{\Delta y}$$

$$\lim_{\substack{\Delta x\to0\\ \Delta y=0}}\frac{u(x+\Delta x,y)+iv(x+\Delta x,y)-u(x,y)-iv(x,y)}{\Delta x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\lim_{\substack{\Delta y\to0\\ \Delta x=0}}\frac{u(x,y+\Delta y)+iv(x,y+\Delta y)-u(x,y)-iv(x,y)}{i\Delta y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\Longrightarrow\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\[2mm] \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases}$$

3) 充要条件

u_x, u_y, v_x, v_y 连续、且满足柯西-黎曼条件

4) 极坐标下的柯西-黎曼条件

$u(x,y)=u(\rho,\varphi)$

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial\rho}\cdot\dfrac{\partial\rho}{\partial x}+\dfrac{\partial u}{\partial\varphi}\dfrac{\partial\varphi}{\partial x}\\[2mm] \dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial\rho}\dfrac{\partial\rho}{\partial y}+\dfrac{\partial u}{\partial\varphi}\dfrac{\partial\varphi}{\partial y}\end{cases}\qquad\begin{cases}\dfrac{\partial\rho}{\partial x}=\dfrac{x}{\rho}=\cos\varphi\\[2mm] \dfrac{\partial\varphi}{\partial x}=\dfrac{1}{-\rho\sin\varphi}\end{cases}$$

$$\begin{cases}\dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial\rho}\dfrac{\partial\rho}{\partial x}+\dfrac{\partial v}{\partial\varphi}\dfrac{\partial\varphi}{\partial x}\\[2mm] \dfrac{\partial v}{\partial y}=\dfrac{\partial v}{\partial\rho}\dfrac{\partial\rho}{\partial y}+\dfrac{\partial v}{\partial\varphi}\dfrac{\partial\varphi}{\partial y}\end{cases}$$

$$\lim_{\substack{\Delta\rho\to0\\ \Delta\varphi=0}}\frac{u(\rho+\Delta\rho,\varphi)-u(\rho,\varphi)+iv(\rho+\Delta\rho,\varphi)-iv(\rho,\varphi)}{\Delta\rho e^{i\varphi}}$$

$$=\lim_{\substack{\Delta\rho=0\\ \Delta\varphi\to0}}\frac{u(\rho,\varphi+\Delta\varphi)-u(\rho,\varphi)+iv(\rho,\varphi+\Delta\varphi)-iv(\rho,\varphi)}{i\rho\Delta\varphi e^{i\varphi}}$$

$$=\lim_{\Delta z\to0}\frac{\Delta f(z)}{\Delta z}=\frac{df(z)}{dz}$$

$$=\frac{\partial u}{\partial\rho}e^{-i\varphi}+i\frac{\partial v}{\partial\rho}e^{-i\varphi}=\frac{1}{i\rho}\frac{\partial u}{\partial\varphi}e^{-i\varphi}+\frac{1}{\rho}\frac{\partial v}{\partial\varphi}e^{-i\varphi}$$

$$\boxed{\begin{cases}\dfrac{\partial u}{\partial\rho}=\dfrac{1}{\rho}\dfrac{\partial v}{\partial\varphi}\\[2mm] \dfrac{\partial v}{\partial\rho}=-\dfrac{1}{\rho}\dfrac{\partial u}{\partial\varphi}\end{cases}}$$

§1.4 解析函数.

1) 定义 ① $f(z)$ 在 $z \in B$ 中, $z=z_0$ 的邻域内处处可导, 称 $f(z)$ 在 z_0 处解析.

② 若 $f(z)$ 在区域B上每点解析, 则称 $f(z)$ 在 B 上解析

2) 性质:

① 在解析区域内的等u线和等v线 相互正交 $\nabla u \cdot \nabla v = 0$ $\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) = 0$

$(u(x,y)=C_1)$ $(v(x,y)=C_2)$

② 实部u和虚部v 均为调和函数

即 $\Delta u = 0$ $\Delta v = 0$

$$\Rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \end{cases}$$

or

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$$

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y}$$

$$\nabla = \vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} \Longrightarrow \nabla\cdot\nabla = \frac{\partial^2}{\partial \rho^2} + \underbrace{\vec{e}_\varphi \cdot \frac{\partial \vec{e}_\rho}{\partial \varphi}}_{1} \cdot \frac{1}{\rho}\frac{\partial}{\partial \rho} + \cancel{\vec{e}_\varphi \cdot \frac{\partial \vec{e}_\varphi}{\partial \varphi}} \frac{1}{\rho}\cdot\frac{\partial}{\partial \varphi} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

$$\frac{\partial \vec{e}_\rho}{\partial \varphi} = \lim_{\Delta\varphi\to 0} \frac{\vec{e}_\rho(\varphi+\Delta\varphi) - \vec{e}_\rho(\varphi)}{\Delta\varphi} = \vec{e}_\varphi$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}$$

$$\frac{\partial \vec{e}_\varphi}{\partial \varphi} = \lim_{\Delta\varphi\to 0} \frac{\vec{e}_\varphi(\varphi+\Delta\varphi) - \vec{e}_\varphi(\varphi)}{\Delta\varphi} = -\vec{e}_\rho$$

③ $f(z)$ 为解析函数

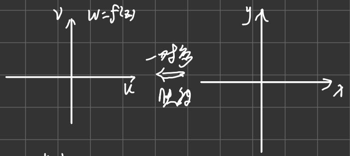
已知u可求v $\longrightarrow$ $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ $v(x,y) = \int dv$ $\begin{cases} ① \text{ 曲线积分} \\ ② \text{ 全微分} \\ ③ \text{ 不定积分} \end{cases}$

已知v可求u

§1.5 平面标量场

$f(z) = z^*$ 非解析函数

§1.6 多值函数.



1) 支点

$z=0$ 为 $\sqrt{z}$ 的一阶支点

$z=0$ 为 $\sqrt[3]{z}$ 的二阶支点

$z=0$ 为 $\ln z$ 的无穷阶支点.

对 $\sqrt{z-a}$, $z=a$ 为一阶支点. $z=\infty$ 亦为支点

对 $\sqrt{z-a}\cdot\sqrt{z-b}$, $z=a$、$z=b$ 均为支点

$\sqrt[3]{(z-a)(z-b)(z-c)}$, $z=a$, $z=b$, $z=c$ 为支点

$\sqrt[5]{(z-a)(z-b)(z-c)}$ $z=a$, $z=b$, $z=c$, $z=\infty$ 为支点

2) 黎曼面 Riemann Surface

将多值函数单值化

补充：

$$\begin{cases} x = \frac{1}{2}(z + z^*) \\ y = \frac{1}{2}(z - z^*) \end{cases}$$

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}$$
$$= \frac{1}{2}\frac{\partial(u+iv)}{\partial x} - \frac{1}{2i}\frac{\partial(u+iv)}{\partial y}$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$
$$= \frac{1}{2}\frac{\partial(u+iv)}{\partial x} + \frac{1}{2i}\frac{\partial(u+iv)}{\partial y}$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{1}{2}i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

§2 复变函数积分

§2.1 复变函数的积分

解析函数、积分与路径无关.

非解析函数，积分或与路径有关.

在$\bar{B}$上分段光滑的曲线L上定义有连续函数$f(z)$

$f(z)$沿L从A到B的路积分：$\int_L f(z)dz = \lim_{\substack{\max|\Delta z_k| \to 0 \\ n \to \infty}} \sum_{k=1}^{n} f(\zeta_k)\Delta z_k$

$$\int_L f(z)dz = \int_L [u(x,y) + iv(x,y)][dx + idy]$$
$$= \int_L (udx - vdy) + i\int_L (vdx + udy)$$

§2.2 柯西定理.

1) 若$f(z)$为闭单连通区域$\bar{B}$上的解析函数. 则 $\oint_L f(z)dz = 0$

$$\oint_L f(z)dz = \oint_L udx - vdy + i\oint_L udy + vdx$$
$$\underline{\underline{\text{Green}}} -\iint_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

Stokes: $\oint E\,d\vec{s} = \iint (\nabla \times E)\,d\vec{S}$

单连通 复连通.

§2.3 不定积分.

若$f(z)$在$\bar{B}$上解析 $\Rightarrow F(z) = \int_{z_0}^{z} f(\zeta)d\zeta \Rightarrow F(z)$在$\bar{B}$上解析，且$F'(z) = f(z)$

$$\lim_{\Delta z \to 0}\frac{F(z+\Delta z) - F(z)}{\Delta z} = \lim_{\Delta z \to 0}\frac{\int_z^{z+\Delta z} f(\zeta)d\zeta}{\Delta z} = f(z)$$

$$I = \oint_l (z-a)^n dz$$

① a 不在 int l $\quad I = 0$

② a 在 int l，
- 1) $n \geq 0 \quad I = 0$
- 2) $n < 0$

$$I = 0 + \oint_C (\sigma e^{i\varphi})^n d(a + \sigma e^{i\varphi})$$

$$= \int_0^{2\pi} \sigma^n e^{in\varphi} \cdot \sigma e^{i\varphi} d\varphi$$

$$= i \int_0^{2\pi} \sigma^{n+1} e^{i(n+1)\varphi} d\varphi \ \underline{\underline{n=-1}} \ i \int_0^{2\pi} d\varphi = 2\pi i$$

$$\underline{\underline{n \neq -1}} \ i \cdot \frac{1}{i(n+1)} \cdot \sigma^{n+1} \left(e^{i(n+1)2\pi} - 1\right)$$

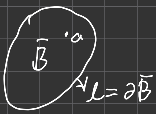
$$= \frac{\sigma^{n+1}}{(n+1)} \left[e^{i(n+1)2\pi} - 1\right] = 0$$

$$\frac{1}{2\pi i} \oint_l \frac{dz}{z-\alpha} = \begin{cases} 0 & (l \text{ 不包围 } \alpha), \\ 1 & (l \text{ 包围 } \alpha). \end{cases}$$

$$\frac{1}{2\pi i} \oint_l (z-\alpha)^n dz = 0 \quad (n \neq -1).$$

§2.4 柯西公式

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_l \frac{f(z)}{z-a} dz}$$



$$f(a) = \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(z)}{z-a} dz = \frac{f(a)}{2\pi i} \int_{C_\varepsilon} \frac{1}{z-a} dz$$

$$\Downarrow$$

$$f(z) = \frac{1}{2\pi i} \oint_l \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

$$f''(z) = \frac{2}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^3} d\zeta$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

模数原理：$f(z)$ 在 $\bar{B}$ 解析，则 $|f(z)|_{max}$ 只能于边界 l 上取到

$$[f(z)]^n = \frac{1}{2\pi i} \oint_l \frac{[f(\zeta)]^n}{\zeta - z} dz \leq \frac{1}{2\pi} \frac{M^n}{\delta} s$$

$$|f(z)| \leq M \left(\frac{s}{2\pi\delta}\right)^{\frac{1}{n}} \leq M$$

如 $|f(\zeta)|$ 在 l 上的极大值为 M，$|\zeta - z|$ 的极小值为 δ，l 的长为 s

§3 幂级数展开

§3.1 复项幂级数

$$\sum_{n=0}^{\infty} w_n = \sum_{n=0}^{\infty} u_n + i\sum_{n=0}^{\infty} v_n$$

$\sum_{n=0}^{\infty} w_n$ 收敛 $\Longleftrightarrow \begin{cases} \sum_{n=0}^{\infty} u_n \\ \sum_{n=0}^{\infty} v_n \end{cases}$ 均收敛

柯西收敛判据

$\exists N \in \mathbb{N}, \forall \varepsilon>0, \forall n>N, p\in \mathbb{N}, \left|\sum_{k=n+1}^{n+p} w_k\right| < \varepsilon \Longleftrightarrow \sum_{n=0}^{\infty} w_n$ 收敛

$\sum_{n=0}^{\infty} |w_n| = \sum_{n=0}^{\infty} \sqrt{u_n^2+v_n^2}$ 收敛 $\Longrightarrow \sum_{n=0}^{\infty} w_n$ 绝对收敛 $\Rightarrow \sum_{n=0}^{\infty} w_n$ 收敛

函数项级数：$\sum_{n=0}^{\infty} w_n(z)$ 若在 B 上所有的点，$\sum_{n=0}^{\infty} w_n(z)$ 都收敛，则 $\sum_{n=0}^{\infty} w_n(z)$ 在 B 上收敛

充要条件：$\exists N\in \mathbb{N}, \forall \varepsilon>0, \forall p\in \mathbb{N}, n>N, z\in B \left|\sum_{k=n+1}^{n+p} w_n(z)\right| < \varepsilon$

若 $\forall \varepsilon>0, \exists N\in \mathbb{N}, \forall n>N, \forall z\in B \left|\sum_{k=1}^{n} w_k(z) - w(x)\right| < \varepsilon \Rightarrow \sum_{n=0}^{\infty} w_n(z) \xrightarrow{B} w(x)$

B 上一致收敛级数每一项 $w_k(z)$ 在 B 上连续，级数和 $w(x)$ 在 B 上连续

若 B 为曲线 l，则级数可沿 l 逐项积分

$$\int_l w(z)\,dz = \int_l \sum_{n=0}^{\infty} w_n(z)\,dz = \sum_{n=0}^{\infty} \int_l w_n(z)\,dz$$

若 $\bar{B}$ 上级数 $\sum_{k=0}^{\infty} w_k(z)$ 一致收敛，$w_k(z)$ 在 $\bar{B}$ 上单值解析，则 $w(z)$ 也为 $\bar{B}$ 中单值解析函数

可逐项求导 $w^{(n)}(z) = \underbrace{\sum_{k=0}^{\infty} w_k^{(n)}(z)}_{\text{在}B\text{内任意一个闭区域中一致收敛}}$

若对 B 上各点 z，$|w_k(z)| \le m_k$，且 $\sum_{k=0}^{\infty} m_k$ 收敛，则 $\sum_{k=0}^{\infty} w_k(z)$ 在 B 上绝对且一致收敛

§3.2 幂级数 (Power Series)

def $\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

a_k, z_0 均为复常数

$\begin{cases} \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| |z-z_0| < 1 & \text{绝对收敛} \\ \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| |z-z_0| > 1 & \text{发散} \end{cases}$

$\begin{cases} \lim_{k\to\infty} \sqrt[k]{|a_k||z-z_0|^k} < 1 & \text{绝对收敛} \\ \lim_{k\to\infty} \sqrt[k]{|a_k||z-z_0|^k} > 1 & \text{发散} \end{cases}$

$R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \frac{1}{\sqrt[k]{|a_k|}}$

收敛半径

① $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ 在收敛圆内部绝对一致收敛

② $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ 可沿收敛圆内部一周周 C_R 上逐项积分

$$\frac{1}{2\pi i}\oint_{C_R}\frac{w(\zeta)}{\zeta-z}d\zeta = a_0 + a_1(z-z_0)+\cdots$$

可将任意多个
为解析函数

③ $w(z)$在收敛圆内为解析函数，④不可能出现奇点，

⑤ 求导或积分不改变收敛半径

§3.3 Taylor级数展开

$(R_1<R)$

$f(z)$在以z_0为圆心的C_R内解析，则 $f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k$，$a_k=\frac{1}{2\pi i}\oint_{C_{R_1}}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta=\frac{f^{(k)}(z_0)}{k!}$

唯一的

§3.4 解析延拓

1) def. 将$f(z)$定义域从b拓到B，找到$F(z)$在B上解析，在$b\subset B$上 $F(z)=f(z)$

解析延拓是唯一的

反证法 假设 $G(z)\equiv f(z)-F(z)$ 存在奇点，证伪

§3.5 洛朗级数展开 (Laurent's Power Series)

1) def.

可含有负幂次项的幂级数展开

除去可去奇点上环域的展开

定理：$f(z)$在环域 $R_2<|z-z_0|<R_1$ 的内部单值解析

则 $f(z)=\sum_{k=-\infty}^{+\infty}a_k(z-z_0)^k$，$a_k=\frac{1}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta \neq \frac{f^{(k)}(z_0)}{k!}$ 多值呢？

- $z=z_0$ 在负幂次下为洛朗级数的奇点时，不一定为原级数奇点
- $a_k \neq \frac{f^{(k)}(z_0)}{k!}$
- 洛朗级数是唯一的.

1) $\frac{\sin z}{z}$ ① $f(z)=\sum_{k=-\infty}^{\infty} a_k z^k$， $a_k=\frac{1}{2\pi i}\oint_c \frac{\sin\zeta}{\zeta^{k+2}}d\zeta$ $|z|>0$，此时所有负幂次项均为0

与 $f(z)$ 相同。由于洛朗展开唯一

② $f(z)=\frac{z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+\cdots$ 只有正幂次的洛朗展开

def $F(z)=\begin{cases} f(z)=\frac{\sin z}{z} & |z|>0 \\ 1 & |z|=0 \end{cases}$ 解析延拓

2) $f(z)=\frac{1}{(z-1)(z-2)}$

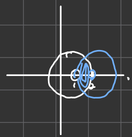
$0<|z-1|<1$

有限负幂次项

$z=1$处Laurent's 展开 $\Rightarrow f(z)=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{z-1-1}-\frac{1}{z-1}=-\frac{1}{1-(z-1)}-\frac{1}{z-1}=-\sum_{n=0}^{\infty}(z-1)^n-(z-1)^{-1}=-\sum_{n=-1}^{\infty}(z-1)^n$

$z=2$处Laurent's 展开 $\Rightarrow f(z)=\frac{1}{z-2}-\frac{1}{1+(z-2)}=(z-2)^{-1}+\sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n=\sum_{n=-1}^{\infty}(-1)^{n+1}(z-2)^n$

$0<|z-2|<1$



3) $f(z)=e^{\frac{1}{z}}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{z}\right)^k=\sum_{k=-\infty}^{0}\frac{1}{(-k)!}z^k$ $(|z|>0)$ 无穷负幂次项

4) $f(z)=e^{\frac{1}{2}x(z-\frac{1}{z})}$ (x为参数)

$e^{\frac{1}{2}x(z-\frac{1}{z})}=e^{\frac{1}{2}xz}\cdot e^{-\frac{1}{2}x\frac{1}{z}}$

$=\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{1}{2}xz\right)^l\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}x\frac{1}{z}\right)^n$ $(0<|z|<\infty)$

$=\sum_{m=-\infty}^{\infty}J_m(x)z^m$

$\triangle$ m阶Bessel函数

§3.6 奇点分类

1) 孤立奇点 & 非孤立奇点

挖去后可展开为洛朗级数（另解析区域） 在z_0邻域内总有不可导点

2) 孤立奇点分类

① 可去奇点 只有正幂次项

$\lim_{z\to z_0}(z-z_0)^m f(z)$ 有限

② 极点 有有限个负幂次项 $\longrightarrow \sum_{k=-m}^{\infty}a_k(z-z_0)^k$ $(0<|z-z_0|<R)$ m为极点z_0的阶。一阶极点称为单极点

③ 本性奇点 有无限个负幂次项

$f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \quad (0<|z-z_0|<R)$

$\lim_{z\to z_0} f(z)$ 由 $z\to z_0$ 路径决定

*无限远点为孤立奇点

$f(z)=\sum_{k=-\infty}^{\infty} a_k z^k \quad (R<|z|<\infty)$

负幂：解析部分
正幂：主要部分 or 无限部分

无正幂项 → 可去奇点
有限正幂项 → 极点
无限正幂项 → 本性奇点

*支点为孤立奇点

对有限阶支点， $m-1$ 阶支点 z_0 ，引入 $\zeta=\sqrt[m]{z-z_0}$ ， ζ 邻域为单叶圆环

$g(\zeta)=\sum_{k=-\infty}^{\infty} a_k\zeta^k = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k/m}$.

无负幂　有限负幂　无限负幂

解析型支点　极点型支点　本性奇点型支点.

§4 留数定理及应用

§4.1 留数定理

1) 定义

$\oint_l f(z)dz=\oint_{l_0} f(z)dz \xlongequal{洛朗展开} \sum_{k=-\infty}^{\infty} a_k\underbrace{\oint_{l_0}(z-z_0)^k dz}_{\text{所有负幂之项积分为0}} = 2\pi i\, a_{-1}$

(-1次项积分且 l_0 包含 z_0 时为 $2\pi i$)

$f(z)$ 在 z_0 的留数（残数），记作 $\text{Res} f(z_0)$

$\oint_l f(z)dz = 2\pi i\, \text{Res} f(z_0)$

2) 留数定理： $f(z)$ 在回路 l 所围区域 B 上除有限个孤立奇点 $b_1,b_2,\dots,b_n$ 外解析，在闭域 $\bar{B}$ 上除奇点外连续

则 $\oint_l f(z)dz = 2\pi i\sum_{j=1}^{n} \text{Res} f(b_j)$

无限远点留数为 $-a_{-1}$ ，即 $\text{Res} f(\infty)$ ， $\text{Res} f(\infty) = -2\pi i \sum_{j=1}^{n} \text{Res} f(z_j)$

$f(z)$ 在全平面上各点(包含无限远点和有限远奇点)的留数和为0

3) 留数计算

① 确定回路所有奇点.

② 对奇点 z_j , $j=1,2,\dots,n$ 作 Laurent 展开，求 a_{-1}

③ 确定奇点类型.

1) 若为可去奇点，则 $a_{-1}=0$

2) 若为极点、判断极点阶数.

m阶极点. $f(z)=a_{-m}(z-z_j)^{-m}+\cdots+a_{-1}(z-z_0)^{-1}+a_0+a_1(z-z_j)+\cdots$

$$\lim_{z\to z_j}(z-z_j)^m f(z)=a_{-m} \Rightarrow (z-z_j)^m f(z)=a_{-m}+\cdots+a_{-1}(z-z_j)^{m-1}+a_0(z-z_0)^m+\cdots$$

$$\Rightarrow \lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_j)^m f(z)\right]=(m-1)!\,a_{-1} \Rightarrow \mathrm{Res}\,f(z_j)=\frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}(z-z_j)^m f(z)$$

3) 若为本性奇点，则考虑 $a_{-1}(z-z_0)^{-1}$

Alternate descriptions [edit]

Let a be a complex number, assume that $f(z)$ is not defined at a but is analytic in some region U of the complex plane, and that every open neighbourhood of a has non-empty intersection with U.

If both

$\lim_{z\to a} f(z)$ and $\lim_{z\to a}\frac{1}{f(z)}$ exist, then a is a removable singularity of both f and $1/f$.

If

$\lim_{z\to a} f(z)$ exists but $\lim_{z\to a}\frac{1}{f(z)}$ does not exist, then a is a zero of f and a pole of $1/f$.

Similarly, if

$\lim_{z\to a} f(z)$ does not exist but $\lim_{z\to a}\frac{1}{f(z)}$ exists, then a is a pole of f and a zero of $1/f$.

If neither

$\lim_{z\to a} f(z)$ nor $\lim_{z\to a}\frac{1}{f(z)}$ exists, then a is an essential singularity of both f and $1/f$.

Another way to characterize an essential singularity is that the Laurent series of f at the point a has infinitely many negative degree terms (i.e., the principal part of the Laurent series is an infinite sum). A related definition is that if there is a point a for which no derivative of $f(z)(z-a)^n$ converges to a limit as z tends to a, then a is an essential singularity of $f(z)$.[1]

The behavior of holomorphic functions near their essential singularities is described by the Casorati–Weierstrass theorem and by the considerably stronger Picard's great theorem. The latter says that in every neighbourhood of an essential singularity a, the function f takes on *every* complex value, except possibly one, infinitely many times. (The exception is necessary, as the function $\exp(1/z)$ never takes on the value 0.)

Model illustrating essential singularity of a complex function 6w=exp(1/(6z))

or 判断是否为pole

$\lim_{z\to z_0}(z-z_0)^m f(z)$ 是否为0

$\frac{1}{\sin z}$ 奇点 $n\pi$. 留数为 $(-1)^n$.

$\mathrm{Res}(\infty)$ 不定.

§4.2 应用留数定理计算实变函数定积分

$$\int_a^b f(x)dx \longmapsto \oint f(z)dz = 2\pi i\sum_{j=1}^{n}\mathrm{Res}\,f(z_j)$$

法1: 映射

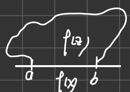
$x\longrightarrow z$

$f(x)\mapsto f(z)$

$a\to b \mapsto$ 回路 l

$$\int_0^{2\pi}R(\cos x,\sin x)\,dx \xlongequal{z=e^{ix}} \oint_{|z|=1}R\left(\frac{z+\frac{1}{z}}{2},\frac{z-\frac{1}{z}}{2}\right)\frac{1}{iz}dz$$

法2: 延拓.



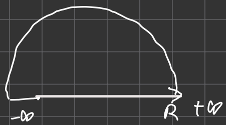
$$\oint_l f(z)dz=\int_a^b f(x)dx+\int_{l_1}f(z)dz+\int_{l_2}f(z)dz\cdots=2\pi i\sum_{j=1}^{n}\mathrm{Res}\,f(z_j)$$

设置 $\int_{l_i}f(z)dz$ —
- 为0
- 简单
- 为 $M\int_a^b f(x)dx$ 形式（倍数形式）

类型 I

$$\int_0^{2\pi} R(\cos x, \sin x)\,dx \qquad z=e^{ix} \quad dx=\frac{1}{iz}dz$$

类型 II



$$\int_{-\infty}^{+\infty} f(x)\,dx + \int_{C_R} \underbrace{f(z)\,dz}_{\to 0} = 2\pi i \sum_{j=0}^{n} \mathrm{Res}\, f(z_j)$$

$z f(z) \xrightarrow[\text{当}z\to\infty]{\text{一致趋于}} 0$，满足 实轴上无奇点． 上半平面 除有限个奇点外解析

$$\left|\int_{C_R} f(z)\,dz\right| = \int_{C_R} |f(z)z| \left|\frac{dz}{z}\right| \le \max_{C_R} |f(z)z| \underbrace{\left|\int_{C_R} \frac{dz}{z}\right|}_{\int_0^{\pi} \frac{1}{e^{i\varphi}} \cdot i e^{i\varphi} d\varphi} = \max_{C_R} |f(z)z| \pi \Rightarrow 0$$

类型 III.

$$\int_0^{\infty} \underbrace{F(x)}_{偶} \cos mx\,dx, \quad \int_0^{\infty} \underbrace{G(x)}_{奇} \sin mx\,dx$$

满足 $z\to\infty$, $F(z)$, $G(z)$ 一致 $\Rightarrow 0$

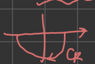
$$\frac{1}{2}\int_{-\infty}^{+\infty} F(x)\cos mx\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} F(x)\frac{e^{imx}+e^{-imx}}{2}dx \overset{F(x)=F(-x)}{=\!=} \frac{1}{2}\int_{-\infty}^{+\infty} F(x)e^{imx}dx + \underbrace{\frac{1}{2}\int_{C_R} F(z)e^{imz}dz}_{0}$$

$$= \pi i \sum_{j=1}^{k} \mathrm{Res}\left(F(z_j)e^{imz_j}\right)$$

$$\frac{1}{2}\int_{-\infty}^{+\infty} G(x)\sin mx\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} G(x)\frac{e^{imx}-e^{-imx}}{2i}dx \overset{G(x)+G(-x)=0}{=\!=} \frac{1}{2i}\int_{-\infty}^{+\infty} G(x)e^{imx}dx + \underbrace{\frac{1}{2i}\int_{C_R} G(z)e^{imz}dz}$$

$$= \pi \sum_{j=1}^{k} \mathrm{Res}\left(G(z)e^{imz}\right)$$

注：$m>0$、$m<0$ 需分情况讨论



$$\left\langle \begin{array}{l} -\pi i \sum_{j=1}^{k} \mathrm{Res}\, f(z_j) \\ -\pi \sum_{j=1}^{k} \mathrm{Res}\, f(z_j) \end{array}\right.$$

约当引理 (Jordan's Lemma)

$f(z)$ 在 $\theta_1 \le \arg z \le \theta_2$, $R<|z|<\infty$ 上连续 且 极限 $\lim_{z\to\infty} f(z) \Longrightarrow 0$

则对 $\forall m>0$, $\lim_{R\to\infty} \int_{C_R(弧)} f(z)e^{imz}dz = 0$

$$\left|\int_{C_R} f(z)e^{imz}dz\right| = \left|\int_{\theta_1}^{\theta_2} f(Re^{i\varphi})e^{imRe^{i\varphi}}dRe^{i\varphi}\right| \le \int_{\theta_1}^{\theta_2} |f(Re^{i\varphi})| e^{-mR\sin\varphi} R\,d\varphi$$

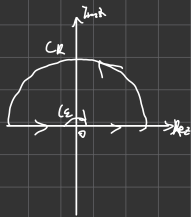
$$\le \int_0^{\pi} |f(Re^{i\varphi})| e^{-mR\sin\varphi} R\,d\varphi \le \max|f(z)| R\int_0^{\pi} e^{-mR\sin\varphi}d\varphi$$

$$= 2\max|f(z)| R\int_0^{\frac{\pi}{2}} e^{-mR\sin\varphi}d\varphi \le 2\max|f(z)| R\int_0^{\frac{\pi}{2}} e^{-mR\frac{2}{\pi}\varphi}d\varphi$$

$$\le \frac{\pi}{mR}\cdot 2\max|f(z)| \Rightarrow 0$$

ep. 实轴上单极点情况

$$\int_0^{\infty}\frac{\sin x}{x}dx = \frac{1}{2i}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}dx = I = -\frac{1}{2i}\int_{C_\varepsilon}\frac{e^{iz}}{z}dz = \frac{\pi}{2}$$



$$\frac{1}{2i}\oint\frac{e^{iz}}{z}dz = \frac{1}{2i}\int_{C_R}\frac{e^{iz}}{z}dz + \frac{1}{2i}\int_{-\infty}^{-\varepsilon}\frac{e^{iz}}{z}dz + \frac{1}{2i}\int_{C_\varepsilon}\frac{e^{iz}}{z}dz + \frac{1}{2i}\int_{\varepsilon}^{+\infty}\frac{e^{iz}}{z}dz = 0$$

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i\sum_{\text{上半平面}}\mathrm{Res}f(z) + \pi i\sum_{\text{实轴上}}\mathrm{Res}f(z)$$

$$= \frac{1}{2i}\int_{\pi}^{0}\frac{e^{i\varepsilon e^{i\varphi}}}{\varepsilon e^{i\varphi}}\varepsilon i e^{i\varphi}d\varphi$$

$$= \frac{i}{2i}\int_{\pi}^{0} e^{i\varepsilon e^{i\varphi}}d\varphi$$

$$= \frac{i}{2i}\int_{\pi}^{0}(1+i\varepsilon e^{i\varphi}+\dots)d\varphi$$

$$= \frac{1}{2i}(-i\pi) = \frac{1}{2}\pi$$

下面简单介绍一下**色散关系**. 对于在上半平面处处解析的函数 $f(z)$，如果当 $z\to\infty$ 时，$f(z)$ 一致地$\to 0$ $(0\leqslant \mathrm{Arg}\,z\leqslant\pi)$，$\alpha$ 为一实数，则按照(4.2.12)，

$$\mathscr{P}\int_{-\infty}^{\infty}\frac{f(x)}{x-\alpha}dx = \pi i\cdot\left\{\frac{f(z)}{z-\alpha}\text{在 }\alpha\text{ 的留数}\right\} = \pi i f(\alpha).$$

分别写出实部和虚部

$$\mathrm{Re}\,f(\alpha) = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,f(x)}{x-\alpha}dx, \qquad (4.2.15)$$

$$\mathrm{Im}\,f(\alpha) = -\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,f(x)}{x-\alpha}dx. \qquad (4.2.16)$$

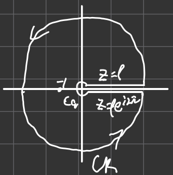
这一对关系式在数学上称为**希尔伯特**(Hilbert,1862—1943)**变换**，在物理上称为色散关系.

§4.3 补充

ep1. $I = \int_0^{\infty} x^{\alpha-1}\frac{1}{1+x}dx \quad (0<\alpha<1)$

$\left(z\cdot z^{\alpha-1}\frac{1}{1+z}\rightrightarrows 0\right)$

对 $f(z) = z^{\alpha-1}\frac{1}{1+z}$，$z=0$ 为支点，$z=-1$ 为单极点



$$\underbrace{\int_{C_R}\frac{z^{\alpha-1}}{1+z}dz}_{0} + \underbrace{\int_{\infty}^{\varepsilon}\frac{\rho^{\alpha-1}e^{i2\pi(\alpha-1)}}{1+\rho e^{i2\pi}}d\rho e^{i2\pi}}_{=-e^{i2\pi\alpha}I} + \underbrace{\int_{C_\varepsilon}\frac{z^{\alpha-1}}{1+z}dz}_{\Downarrow\text{Taylor}} + \underbrace{\int_{\varepsilon}^{+\infty}\frac{\rho^{\alpha-1}}{1+\rho}d\rho}_{I} = (1-e^{i2\pi\alpha})I$$

$$\int_{C_\varepsilon} z^{\alpha-1}dz \Rightarrow z^{\alpha} \Rightarrow 0$$

$$= 2\pi i\,\mathrm{Res}f(-1)$$

$$= 2\pi i(-1)^{\alpha-1}$$

$$\frac{e^{-i\pi\alpha}(1-e^{i2\pi\alpha})I}{2i} = \pi e^{i\pi(\alpha-1)}\cdot e^{-i\pi\alpha}$$

$$-\sin\pi\alpha\, I = -\pi$$

$$I = \frac{\pi}{\sin\pi\alpha}$$

$$I = \int_0^{\infty}\frac{x^{\alpha-1}}{1-x}dx = \pi\tan^{-1}\pi\alpha = \pi\frac{\cos\pi\alpha}{\sin\pi\alpha}$$

计算 $\int_0^{\infty} f(x)\ln x\,dx$

考虑 $\oint f(z)\ln^2 z\,dz = 4\pi i\int_R^{\delta} f(x)\ln x\,dx - 4\pi^2\int_R^{\delta} f(x)dx$

$\int_0^{+\infty} f(x)dx$**型积分**

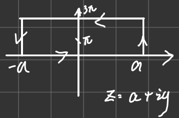
此时f不是偶函数，下面我们不加证明的给出两个定理

定理5. 设$f(z)$在$\mathbb{C}\setminus[0,+\infty)$上除去点$a_1,a_2,\cdots,a_n$外全纯,连续到正实轴，若 $\lim\limits_{z\to\infty}|z|^p f(z) = 0 (p>0)$，则

$$\int_0^{+\infty} f(x)x^{p-1}dx = \frac{2\pi i}{1-e^{i2p\pi}}\sum_{k=1}^{n}\mathrm{Res}(z^{p-1}f(z),a_k)$$

其中$z^{p-1} = |z|^{p-1}e^{i(p-1)\arg z}, 0<\arg z<2\pi.$

ep3. $I=\int_{-\infty}^{+\infty}\frac{e^{\alpha x}}{1+e^{x}}dx \quad 0<\alpha<1$



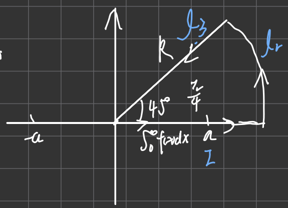
ep4. 计算菲涅尔积分

(Fresnel integral)

$I_1=\int_0^{\infty}\sin x^2dx \quad I_2=\int_0^{\infty}\cos x^2dx$

$f(z)=e^{iz^2}$

$\int_0^{\infty}f(x)dx=I_2+iI_1$



$\oint f(z)dz=0=I+\int_{l_r,\,R\to\infty}f(z)dz+\int_{l_3,\,R\to\infty}f(z)dz$

$=I+\int_0^{\frac{\pi}{4}}e^{iR^2e^{i2\varphi}}iRe^{i\varphi}d\varphi$

$+\int_{\infty}^{0}e^{i\rho^2e^{\frac{i\pi}{2}}}\cdot e^{\frac{i\pi}{4}}d\rho$

$e^{\frac{i\pi}{4}}\int_0^{\infty}e^{-\rho^2}d\rho=e^{\frac{i\pi}{4}}\frac{\sqrt{\pi}}{2}$

$\left|\int_0^{\pi/4}e^{iR^2\cos2\varphi-R^2\sin2\varphi}e^{i\varphi}iRd\varphi\right|<\int_0^{\pi/4}e^{-R^2\sin2\varphi}Rd\varphi$

$<\int_0^{\pi/4}e^{-\frac{4}{\pi}R^2\varphi}Rd\varphi$

$=\frac{-\pi}{4R}(e^{-R^2}-1)\to0$

$\Rightarrow 0=I+e^{\frac{i\pi}{4}}\cdot\frac{\sqrt{\pi}}{2}$

$I=\frac{\sqrt{2\pi}}{4}-i\frac{\sqrt{2\pi}}{4}$

§5 傅立叶变换 (Fourier Transformation)

若 $l\to\infty$ $f(x)=f(x+2l)$ 即非周期函数.

则可先展开，再令 $l\to\infty$.

$f(x)=a_0+\sum_{\omega}a_\omega\cos\omega x+\sum_{\omega}b_\omega\sin\omega x$

$a_0=\frac{1}{2l}\underbrace{\int_{-l}^{l}f(x)dx}_{\text{finite}}\xrightarrow{l\to\infty}0$

$a_\omega=\lim_{l\to\infty}\frac{1}{l}\int_{-l}^{l}f(\xi)\cos\omega\xi d\xi$

$b_\omega=\lim_{l\to\infty}\frac{1}{l}\int_{-l}^{l}f(\xi)\sin\omega\xi d\xi$

$\sum_{\omega}\lim_{l\to\infty}\frac{1}{l}\int_{-l}^{l}f(\xi)\cos\omega\xi d\xi\cdot\cos\omega x$

$=\frac{1}{\pi}\sum_{\omega}\lim_{l\to\infty}\int_{-l}^{l}f(\xi)\cos\omega\xi d\xi\cos\omega x\underbrace{\Delta\omega}_{d\omega}$

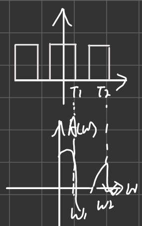
$=\frac{1}{\pi}\sum_{\omega}\int_0^{\infty}\int_{-\infty}^{+\infty}f(\xi)\cos\omega\xi d\xi\cos\omega xd\omega$

$=\frac{1}{\pi}\sum_{\omega}\int_0^{\infty}A(\omega)\cos\omega xd\omega$

$\Downarrow$

$f(x)=\int_0^{\infty}A(\omega)\cos\omega xd\omega+\int_0^{\infty}B(\omega)\sin\omega xd\omega$

$$\begin{cases}A(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}f(\xi)\cos\omega\xi d\xi\\ B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)\sin\omega\xi d\xi\end{cases}$$

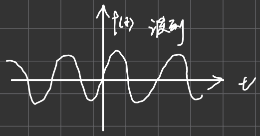


频谱.

离散

ep.

$$f(t)=\begin{cases}0 & t<-\frac{N2\pi}{\omega_0}\\ A\sin\omega_0 t & -N\frac{2\pi}{\omega_0}<t<N\frac{2\pi}{\omega_0}\\ 0 & t>N\frac{2\pi}{\omega_0}\end{cases}$$

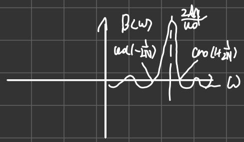


$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

$$=\frac{2}{\pi}\int_{0}^{\infty} A\sin\omega_0 t\sin\omega t\,dt$$

$$=\frac{A}{\pi}\int_{0}^{N\frac{2\pi}{\omega_0}}\left[\cos(\omega_0 t-\omega t)-\cos(\omega_0 t+\omega t)\right]dt$$

$$=\frac{2A\omega_0}{\pi(\omega^2-\omega_0^2)}\sin\left(N2\pi\frac{\omega}{\omega_0}\right)$$



复数形式 Fourier transformation

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(\omega)e^{i\omega x}d\omega$$

$$F(\omega)=\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(\xi)e^{-i\omega\xi}d\xi$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(\xi)e^{-i\omega\xi}e^{i\omega x}\,d\omega\,d\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\xi)d\xi\int_{-\infty}^{+\infty}e^{i\omega(x-\xi)}d\omega \qquad \frac{1}{i(x-\xi)}e^{i\omega(x-\xi)}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\xi)d\xi\cdot\underbrace{2\pi\delta(x-\xi)}_{\substack{\infty\ \ x=\xi\\ 0\ \ x\neq\xi}}=\int_{-\infty}^{+\infty}f(\xi)\delta(x-\xi)d\xi=\int_{x-\varepsilon}^{x+\varepsilon}f(\xi)\delta(x-\xi)d\xi=f(x)\int_{x-\varepsilon}^{x+\varepsilon}\delta(x-\xi)d\xi=f(x)$$

傅立叶变换性质

1) 导数定理:

$$\tilde{F}[f(x)]=F(\omega)$$

$$\tilde{F}[f'(x)]=i\omega F(\omega)$$

2) 积分定理:

$$\tilde{F}\left[\int_{-\infty}^{x}f(\xi)d\xi\right]=\frac{F(\omega)}{i\omega}$$

3) 延迟定理

$$\tilde{F}[f(x-x_0)]=e^{-i\omega x_0}F(\omega)$$

4) 位移定理

$$\tilde{F}[e^{i\omega_0 x}f(x)]=F(\omega-\omega_0)$$

5) 相似定理

$$\tilde{F}[f(ax)]=\frac{1}{a}F\left(\frac{\omega}{a}\right)$$

设 k_0 始

6) 卷积定理

$$\tilde{F}\left[\int_{-\infty}^{+\infty}f_1(\xi)f_2(x-\xi)d\xi\right]=2\pi F_1(\omega)F_2(\omega)$$

eg. $y''(x)+k^2y(x)=A\sin\omega_0 x$

$$\Downarrow$$

$$-\omega^2Y(\omega)+k^2Y(\omega)=\frac{A}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{i\omega_0 x}-e^{-i\omega_0 x}}{2i}\cdot e^{-i\omega x}dx$$

$$=\frac{A}{\sqrt{2\pi}}\cdot\frac{1}{2i}\int_{-\infty}^{+\infty}\left[e^{ix(\omega_0-\omega)}-e^{-ix(\omega_0+\omega)}\right]dx$$

$$=\frac{A}{\sqrt{2\pi}}\cdot\frac{1}{2i}\cdot 2\pi[\delta(\omega_0-\omega)-\delta(\omega_0+\omega)]$$

$$=\tilde{F}[A\sin\omega_0 x]$$

$$Y(\omega)=\frac{\tilde{F}[A\sin\omega_0 x]}{k^2-\omega^2}$$

$$y(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}Y(\omega)e^{i\omega x}d\omega$$

$$=C\frac{i}{k^2-\omega^2}A\sin\omega_0 x$$

三维空间的Fourier transformation

$$f(\vec{r})=f(x,y,z)=\frac{1}{(2\pi)^{3/2}}\iiint F(\vec{k})e^{i\vec{k}\cdot\vec{r}}d^3\vec{k}$$

$$F(\vec{k})=\frac{1}{(2\pi)^{3/2}}\iiint f(\vec{r})e^{-i\vec{k}\cdot\vec{r}}d^3\vec{r}$$

$$=\frac{1}{(2\pi)^{3/2}}\iiint f(x,y,z)e^{-ik_x x-ik_y y-ik_z z}dxdydz$$

§5.3 $\delta(x)$函数

Question1: 证明一个质点与一个均匀球壳间引力等于该质点与球壳质量相同的位于球心处的质点引力相同

Question2: Submarine 是否对周围引力场有改变

1) 引入. 理想模型

质点, 点电荷, 瞬时力

2) 函数.

$$\delta(x-x_0)=\begin{cases}0 & x\neq x_0\\ \infty & x=x_0\end{cases}$$

$$\int_a^b\delta(x-x_0)dx=\begin{cases}1 & a<x_0<b\\ 0 & b<x_0 \text{ 或 } a>x_0\end{cases}$$

$\delta(x)$量纲 $[\delta(x)]=1/[x]$

性质 ① $\delta(x)$为偶函数

$$\delta(x)=\delta(-x)$$

② 积分是Step函数

$$\int_{-\infty}^{x}\delta(x-x_0)dx=\begin{cases}0 & x<x_0\\ 1 & x>x_0\end{cases}$$

Def $H(x)=\int_{-\infty}^{x}\delta(x-x_0)dx$

$$\frac{dH(x)}{dx}=\delta(x-x_0)$$

③ 选择性

$$\int_a^b f(x)\delta(x-x_0)dx=f(x_0)\quad \text{when } x_0\in(a,b)$$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon}f(x)\delta(x-x_0)dx=f(x_0)\int_{x_0-\varepsilon}^{x_0+\varepsilon}\delta(x-x_0)dx=f(x_0)$$

④ $\delta[\varphi(x)]=\begin{cases}0 & \varphi(x)\neq 0\\ \infty & \varphi(x)=0\end{cases}$

$$\varphi(x)=A_k\prod_k(x-x_k)$$

$$\delta[\varphi(x)]=\sum_k C_k\delta(x-x_k)$$

$$\int_{x_k-\varepsilon}^{x_k+\varepsilon}\delta[\varphi(x)]dx=\int_{x_k-\varepsilon}^{x_k+\varepsilon}\sum_l C_l\delta(x-x_l)dx=\int_{x_k-\varepsilon}^{x_k+\varepsilon}C_k\delta(x-x_k)dx$$

$$=\int_{x_k-\varepsilon}^{x_k+\varepsilon}\frac{\delta[\varphi(x)]}{\varphi'(x)}d\varphi(x)=\frac{1}{|\varphi'(x_k)|}\qquad =C_k$$

$\delta(x^2-a^2)=\frac{\delta(x-a)}{2|a|}+\frac{\delta(x+a)}{2|a|}$

$\delta(x^2)=\frac{\delta(x)}{|x|}$

⑤ 傅立叶变换.

$$\delta(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}C(\omega)e^{i\omega x}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega x}d\omega$$

$$C(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}$$

3) $\delta(x)$ 可由多种基本函数的极限来表示.

① $\delta(x)=\lim_{k\to\infty}\frac{1}{2\pi}\int_{-k}^{k}e^{i\omega x}d\omega$

$=\lim_{k\to\infty}\frac{1}{ix}\frac{1}{2\pi}\cdot e^{i\omega x}\Big|_{-k}^{k}$

$=\lim_{k\to\infty}\frac{1}{2\pi ix}(e^{ikx}-e^{-ikx})$

$=\lim_{k\to\infty}\frac{\sin kx}{\pi x}$

② $\delta(x)=\lim_{\varepsilon\to0}\frac{1}{2\pi}\left[\int_{-\infty}^{0}e^{(\varepsilon+ix)\omega}d\omega+\int_{0}^{\infty}e^{(-\varepsilon+ix)\omega}d\omega\right]$

$=\frac{1}{2\pi}\lim_{\varepsilon\to0}\left[\frac{1}{\varepsilon+ix}+\frac{1}{\varepsilon-ix}\right]$

$=\frac{1}{2\pi}\lim_{\varepsilon\to0}\frac{2\varepsilon}{\varepsilon^2+x^2}$

$=\frac{1}{\pi}\lim_{\varepsilon\to0}\frac{\varepsilon}{\varepsilon^2+x^2}$

③ $\delta(x)=\lim_{\sigma\to0}\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$

4) 高维.

$$\delta(\vec r-\vec r_0)=\begin{cases}0 & \vec r\neq\vec r_0\\ \infty & \vec r=\vec r_0\end{cases}$$

$$\int\delta(\vec r-\vec r_0)\,d^3\vec r=1$$

$\delta(\vec r-\vec r_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ 笛卡子

$\delta(\vec r-\vec r_0)=\frac{1}{\rho}\delta(\rho-\rho_0)\delta(\varphi-\varphi_0)\delta(z-z_0)$ 柱坐标

$\delta(\vec r-\vec r_0)=\frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)$ 球坐标

ep.

§6. 拉普拉斯变换 (Laplace transformation) (LT)

§6.1

For Fourier Transformation

$$\tilde F[y(x)]=\bar y(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y(x)e^{-i\omega x}dx$$

$$y(x)=\tilde F^{-1}[\bar y(\omega)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar y(\omega)e^{i\omega x}d\omega$$

$y''(x)+py'(x)+qy(x)=f(x)$

$(i\omega)^2\bar y(\omega)+(i\omega)p\bar y(\omega)+q\bar y(\omega)=\bar f(\omega)$

$\bar y(\omega)=\bar f(\omega)/(-\omega^2+i\omega p+q)$

For Laplace Transformation

Def $\begin{cases}\varphi(t), & t>0\\ 0, & t<0\end{cases}$ or $\varphi(t)H(t)$ $\quad H(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$

$\mathcal{L}[\varphi(t)]=\bar\varphi(p)=\int_0^{\infty}\varphi(t)e^{-pt}dt$ $\quad \varphi(t)\risingdotseq\bar\varphi(p)$

$\varphi(t)=\mathcal{L}^{-1}[\bar\varphi(p)]=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar\varphi(p)e^{pt}dp$ $\quad \bar\varphi(p)\fallingdotseq\varphi(t)$

def $p=\sigma+i\omega$

$$\mathcal{L}(1)=\mathcal{L}[H(t)]=\int_0^\infty e^{-pt}dt=\frac{1}{p}$$

$$\mathcal{L}(t^n)=\mathcal{L}(t^nH(t))=\int_0^\infty t^n e^{-pt}dt=\frac{1}{-p}\int_0^\infty t^n de^{-pt}=\frac{1}{-p}\left[t^ne^{-pt}\Big|_0^\infty-n\int_0^\infty e^{-pt}t^{n-1}dt\right]=\frac{n}{p}\int_0^\infty e^{-pt}t^{n-1}dt=\frac{n}{p}\mathcal{L}[t^{n-1}]$$

$$=\frac{n(n-1)}{p^2}\mathcal{L}[t^{n-2}]=\cdots=\frac{n!}{p^{n+1}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{p}\right]=H(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{p^{n+1}}\right]=\frac{t^n}{n!}$$

$$\mathcal{L}[e^{st}]=\int_0^\infty e^{st}e^{-pt}dt=\frac{1}{s-p}\int_0^\infty e^{(s-p)t}dt=\frac{1}{p-s}$$

$$\mathcal{L}[t^ne^{st}]=\int_0^\infty e^{(s-p)t}t^ndt=\frac{n!}{(p-s)^{n+1}}$$

性质: ① $\mathcal{L}[c_1\varphi_1(t)+c_2\varphi_2(t)]=c_1\mathcal{L}[\varphi_1(t)]+c_2\mathcal{L}[\varphi_2(t)]$ 线性性.

$$\mathcal{L}(\sin\omega t)=\mathcal{L}\left[\frac{1}{2i}(e^{i\omega t}-e^{-i\omega t})\right]=\frac{1}{2i}\left[\frac{1}{p-i\omega}-\frac{1}{p+i\omega}\right]=\frac{\omega}{p^2+\omega^2}$$

$$\mathcal{L}(\cos\omega t)=\mathcal{L}\left[\frac{1}{2}(e^{i\omega t}+e^{-i\omega t})\right]=\frac{1}{2}\left[\frac{1}{p-i\omega}+\frac{1}{p+i\omega}\right]=\frac{p}{p^2+\omega^2}$$

② 导数定理

$$\mathcal{L}\left(\frac{d\varphi(t)}{dt}\right)=\int_0^\infty \frac{d\varphi(t)}{dt}e^{-pt}dt=\varphi(t)e^{-pt}\Big|_0^\infty+p\int_0^\infty\varphi(t)e^{-pt}dt=-\varphi(0)+p\mathcal{L}[\varphi(t)]$$

$$\Rightarrow \mathcal{L}\left[\frac{d\varphi(t)}{dt}\right]=p\mathcal{L}[\varphi(t)]-\varphi(0)$$

or

$$\bar{\varphi}'(p)=p\bar{\varphi}(p)-\varphi(0)$$

$$\mathcal{L}[\varphi^{(n)}(t)]=p^n\mathcal{L}[\varphi(t)]-p^{n-1}\varphi(0)-p^{n-2}\varphi'(0)-\cdots-\varphi^{(n-1)}(0)$$

③ 积分定理.

$$\varphi(t)=\int_0^t\psi(\tau)d\tau \quad (t>0)$$

$$\mathcal{L}\left[\int_0^t\psi(\tau)d\tau\right]=\int_0^\infty\int_0^t\psi(\tau)d\tau\, e^{-pt}dt=\frac{1}{-p}\int_0^\infty\int_0^t\psi(\tau)d\tau\, de^{-pt}=-\frac{1}{p}\left[\int_0^t\psi(\tau)d\tau\, e^{-pt}\Big|_0^\infty-\int_0^\infty e^{-pt}\psi(t)dt\right]$$

$$=\frac{1}{p}\mathcal{L}[\psi(t)]$$

$$\begin{cases} y''(t)+ay'(t)+by(t)=f(t) \\ y(0)=k_1,\ y'(0)=k_2 \end{cases}$$

$$\Rightarrow [p^2\bar{y}(p)-py(0)-y'(0)]+a[p\bar{y}(p)-y(0)]+b\bar{y}(p)=\bar{f}(p)$$

$$(p^2+ap+b)\bar{y}(p)=\bar{f}(p)+(p+a)k_1+k_2$$

$$\bar{y}(p)=\frac{\bar{f}(p)+(p+a)k_1+k_2}{p^2+ap+b}$$

$$y(t)=\mathcal{L}^{-1}[\bar{y}(p)]$$

$$\bar{\varphi}(p) = \int_0^\infty e^{-pt}\varphi(t)dt$$

$$\frac{d\bar{\varphi}(p)}{dp} = -\int_0^\infty e^{-pt}t\varphi(t)dt$$

$$\mathcal{L}[t\varphi(t)] = -\frac{d\bar{\varphi}(p)}{dp}$$

④ $\mathcal{L}[t^n\varphi(t)] = (-1)^n \frac{d^n\bar{\varphi}(p)}{dp^n}$

$\mathcal{L}[e^{st}] = \frac{1}{p-s}$

$\mathcal{L}[te^{st}] = \frac{1}{(p-s)^2}$

$\vdots$

$\mathcal{L}[t^n e^{st}] = \frac{n!}{(p-s)^{n+1}}$

$$\begin{cases} ty''(t) + (1-n)y'(t) + y(t) = 0 \\ y'(0) = 0,\ y(0) = 0 \end{cases}$$

$\mathcal{L}(y''(t)) = p^2\bar{y}(p) - py(0) - y'(0)$

$\mathcal{L}(ty''(t)) = -\frac{d}{dp}[p^2\bar{y}(p)]$

$-\frac{d}{dp}[p^2\bar{y}(p)] + (1-n)p\bar{y}(p) + \bar{y}(p) = 0$

$-2p\bar{y}(p) - p^2\frac{d}{dp}\bar{y}(p) + (1-n)p\bar{y}(p) + \bar{y}(p) = 0$

$\frac{d\bar{y}(p)}{\bar{y}(p)} = \frac{-2p+(1-n)p+1}{p^2} = \frac{1-(n+1)p}{p^2}$

$\ln\bar{y}(p) = -\frac{1}{p} - \ln p^{n+1} + C$

$\bar{y}(p) = \frac{\tilde{C}}{p^{n+1}}e^{-\frac{1}{p}}$

$y(t) = \mathcal{L}^{-1}[\bar{y}(p)]$

§6.2 Laplace 变换反演

$\bar{\varphi}(p) = \frac{1}{p}$ $\quad \varphi(t) = H(t)$

$\bar{\varphi}(p) = \frac{1}{p^{n+1}}$ $\quad \varphi(t) = \frac{t^n}{n!}$

$\bar{\varphi}(p) = \frac{1}{(p-s)^{n+1}}$ $\quad \varphi(t) = \frac{t^n}{n!}e^{st}$

$\bar{\varphi}(p) = \frac{\omega}{p^2+\omega^2}$ $\quad \varphi(t) = \sin\omega t$

$\bar{\varphi}(p) = \frac{p}{p^2+\omega^2}$ $\quad \varphi(t) = \cos\omega t$

$\bar{\varphi}(p) = \frac{1}{ap^2+bp+c}$ $\quad \varphi(t) = \frac{2}{\sqrt{4ac-b^2}}e^{-\frac{b}{2a}t}\sin\frac{\sqrt{4ac-b^2}}{2a}t$

$= \frac{1}{a(p+\frac{b}{2a})^2+\sqrt{c-\frac{b^2}{4a}}}$

辅助反演定理

$\bar{\varphi}(p) \doteq \varphi(t)$

① 延迟定理

$e^{-\tau p}\bar{\varphi}(p) \doteq \varphi(t-\tau)$

$\frac{1}{\sqrt{p}} \doteq \frac{1}{\sqrt{\pi t}}$

$\frac{e^{-\tau p}}{\sqrt{p}} = \frac{1}{\sqrt{\pi(t-\tau)}}$

② 位移定理

$\bar{\varphi}(p+\lambda) \doteq e^{-\lambda t}\varphi(t)$ $\qquad \bar{\varphi}(p-\lambda) = e^{\lambda t}\varphi(t)$

③ 卷积定理

$\bar{\varphi}_1(p)\bar{\varphi}_2(p) \doteq \int_0^t \varphi_1(\tau)\varphi_2(t-\tau)d\tau = \int_0^t \varphi_2(\tau)\varphi_1(t-\tau)d\tau$

$$\int_0^\infty\int_0^t \varphi_1(\tau)\varphi_2(t-\tau)d\tau\, e^{-pt}dt = \int_0^\infty \varphi_1(\tau)d\tau\int_\tau^\infty \varphi_2(t-\tau)e^{-pt}dt = \int_0^\infty \varphi_1(\tau)d\tau\int_0^\infty \varphi_2(t-\tau)e^{-p(t-\tau)}e^{-p\tau}d(t-\tau)$$

$$= \bar{\varphi}_2(p)\int_0^\infty \varphi_1(\tau)e^{-p\tau}d\tau$$

黎曼-梅林反演公式

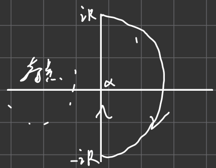
已知 $\overline{\varphi(p)}$，求 $\varphi(t)$

$$\varphi(t)=\frac{1}{2\pi i}\int_b \overline{\varphi(p)}\,e^{pt}dp$$

表示奇点在左侧

实部足够大

虚部从 $-\infty$ 积到 $+\infty$



$$\overline{\varphi}(p) = -\frac{1}{2\pi i}\oint \frac{\overline{\varphi}(\xi)}{\xi-p}d\xi$$

$$= -\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\alpha-iR}^{\alpha+iR}\frac{\overline{\varphi}(\xi)}{\xi-p}d\xi - \underbrace{\lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_R}\frac{\overline{\varphi}(\xi)}{\xi-p}d\xi}$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\overline{\varphi}(\xi)}{p-\xi}d\xi$$

$$\left\{\begin{aligned}&|\overline{\varphi}(\xi)| < \frac{1}{\xi-p}\\ &\xi-p\sim Re^{i\theta}\end{aligned}\right.$$

$\Downarrow$

原式 $=\lim_{R\to\infty}\frac{1}{2\pi i}\int_0 \frac{1}{R}\cdots d\theta \longrightarrow 0$

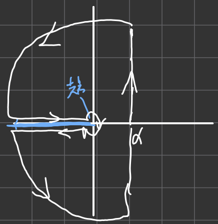
$\Downarrow$ 同时逆变换

$$\varphi(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\overline{\varphi(\xi)}\,e^{\xi t}d\xi$$

$$= \frac{1}{2\pi i}\int_b \overline{\varphi(p)}\,e^{pt}dp$$

$\overline{\varphi}(p) = \frac{1}{\sqrt{p}}$ 求 $\varphi(t)$

$$\varphi(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\sqrt p}e^{pt}dp$$



$$\oint \frac{1}{\sqrt p}e^{pt}dp = 0 = \varphi(t) + \left[\underbrace{\int_{C_{R_1}}\frac{1}{\sqrt p}e^{pt}dp}_{\to 0} + \int_{-\infty}^{-\varepsilon}\frac{1}{\sqrt p}e^{pt}dp + \underbrace{\int_{C_\varepsilon}\frac{1}{\sqrt p}e^{pt}dp}_{\to 0} + \int_{-\varepsilon}^{-\infty}\frac{1}{\sqrt p}e^{pt}dp + \underbrace{\int_{C_{R_2}}\frac{1}{\sqrt p}e^{pt}dp}_{\to 0}\right]\frac{1}{2\pi i}$$

$$\Rightarrow \varphi(t) = -\frac{2}{2\pi i}\int_{-\infty}^{-\varepsilon}\frac{1}{\sqrt p}e^{pt}dp = -\frac{2}{2\pi i}\int_{-\infty}^{0}\frac{1}{\sqrt p}e^{pt}dp = \frac{2}{2\pi i}\int_0^{-\infty}\frac{1}{\sqrt p}e^{pt}dp = \frac{1}{\pi i}\int_0^{\infty}\frac{1}{\sqrt p\, i}e^{-pt}d(p)$$

$$= \frac{2}{\pi}\int_0^{\infty}e^{-(\sqrt p)^2 t}d(\sqrt p)$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\sqrt{\frac{\pi}{t}} = \sqrt{\frac{1}{\pi t}}$$

§7 数学物理定解方法.

共性：二阶线性偏微分方程

(a) Laplace 方程、Poisson 方程.

静电势 & 引力

(b) 波动方程

波的传播.

(c) 热传导方程.

热传导 & 扩散

(d) Maxwell 方程.

电磁场的变化

(e) Schrödinger 方程. Dirac 方程.

微观物质运动.

……

pde: $u \equiv u(\vec{r}, t)$

$\vec{r} \equiv (x, y, z)$

$\cdots \frac{\partial^2 u}{\partial r^2} + \cdots = \cdots$

空间 N 维 + 时间 = $(N+1)$ 维 "空间" $\underbrace{}_{\text{时空}}$

二阶偏微分

$\underbrace{\hat{L}\, u(x_0, x_1 \cdots x_N)}_{\text{物理场量}} = \underbrace{f(x_0 \cdots x_N)}_{\text{非齐次项}}$ 若 $f=0$ 则为齐次方程

$$\hat{L} = \sum_{i,j} a_{ij}(x_0 \cdots x_N) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x_0 \cdots x_N) \frac{\partial}{\partial x_i} + c(x_0 \cdots x_N)$$

线性算子: $\hat{L}_1, \hat{L}_2$ $\quad \alpha(\hat{L}_1 + \hat{L}_2) = \alpha\hat{L}_1 + \alpha\hat{L}_2$

$(\alpha+\beta)\hat{L}_1 = \alpha\hat{L}_1 + \beta\hat{L}_1$

$(\alpha\beta)\hat{L}_1 = \alpha(\beta\hat{L}_1)$

若为齐次方程

u_1, u_2 的解 $\quad \hat{L} u_1 = \hat{L} u_2 = 0$

$u = c_1 u_1 + c_2 u_2$

$\hat{L} u = c_1 \hat{L} u_1 + c_2 \hat{L} u_2$

3D欧氏空间

(a) $a_{00}=1$, $a_{11}=a_{22}=a_{33}=-a^2$. 其它$=0$

$$\hat{L}=\frac{\partial^2}{\partial t^2}-a^2\nabla^2$$ 双曲算子 (波动)

波动 Eq: $u_{tt}-a^2\nabla^2 u=f$

a — 波速

$\begin{cases} t\to -t & \text{时间反演对称（可逆）} \\ \vec{r}\to-\vec{r} & \text{空间对称} \end{cases}$

(b) $b_0=1$, $a_{11}=a_{22}=a_{33}=-a^2$, 其余为0

$$\hat{L}=\frac{\partial}{\partial t}-a^2\nabla^2.$$ 抛物算子 (输运)

$u_t-a^2\nabla^2 u=f$

a - 输运系数

$\begin{cases} \vec{r}\to-\vec{r} & \text{空间对称.} \\ t\not\to -t & \text{时间不可逆} \end{cases}$

(c) $a_{11}=a_{22}=a_{33}=1$, 其余为0 无∂_t (即t) (不需初始条件)

$\hat{L}=\nabla^2$ Laplace算子

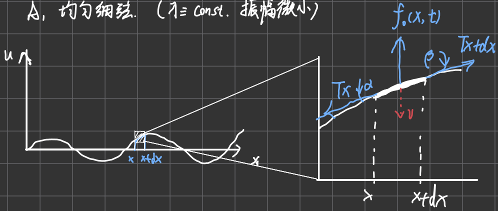
$\nabla^2 u=f$ Poisson方程 (稳定场) 椭圆算子

$f=0 \to \nabla^2 u=0$ Laplace方程 (无源稳定场)

Eg.

A. 波动方程

A. 均匀细弦. ($\lambda\equiv$ const. 振幅微小)



x向

$T_x\cos\alpha - T_{x+dx}\cos\beta = 0$

$\Big\downarrow \alpha,\beta\to 0$

$T_x = T_{x+dx}\approx T$

u向

$f_0(x,t)dx + T_{x+dx}\sin\beta - T_x\sin\alpha = \lambda dx\cdot u_{tt}$

$\Big\downarrow \alpha,\beta\to 0$

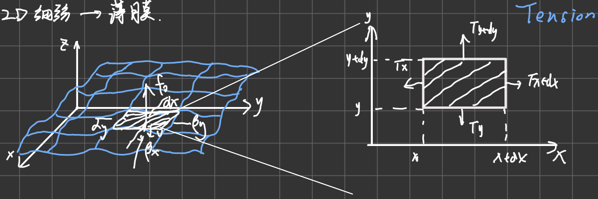
$\begin{cases} \sin\alpha\approx\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_x \\ \sin\beta\approx\tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x+dx} \end{cases}$

$T\left(\left.\frac{\partial u}{\partial x}\right|_{x+dx}-\left.\frac{\partial u}{\partial x}\right|_x\right)+f_0(x,t)dx=\lambda dx\, u_{tt}$

$\overset{dx\to 0}{=\!=}\quad T\frac{\partial^2 u}{\partial x^2}\cancel{dx}+f_0(x,t)\cancel{dx}=\lambda\cancel{dx}\,u_{tt}$

$\Rightarrow\quad u_{tt}-a^2u_{xx}=f(x,t)$ $a^2=T/\lambda$, $f=f_0/\lambda$

2D 细弦 → 薄膜.



$$T_x \approx T_{x+dx} \approx T_y \approx T_{y+dy} = T$$

$$\underbrace{(T\sin\beta_x - T\sin\alpha_x)\,dy}_{①} \approx T u_{xx}\,dx\,dy$$

$$\underbrace{(T\sin\beta_y - T\sin\alpha_y)\,dx}_{②} \approx T u_{yy}\,dx\,dy$$

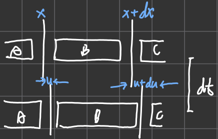
$$② + ① + f_0(x,y,t)\,dxdy = \sigma\,dxdy\,u_{tt}$$

$$\Longrightarrow \quad T(u_{xx}+u_{yy}) + f_0(x,y,t) = \sigma u_{tt}$$

$$u_{tt} - a^2\nabla^2 u = f(x,y,t)$$

$$a = \sqrt{T/\sigma} \qquad f = f_0/\sigma$$

A2 杆的纵振动



B 相对拉伸 $\frac{du}{dx} = u_x$

B 两边张力 $p(x,t)S$, $p(x+dx,t)S$

$$\rho \cdot S \cdot dx\, u_{tt} = [p(x+dx,t) - p(x,t)]S$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial p}{\partial x}$$

由 Hooke 定律 $\quad p\big|_x = E\frac{\partial u}{\partial x}$ （E：杨氏模量）

$$\Longrightarrow \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$$

$$a = \sqrt{E/\rho}$$

3D 纵波:

$$\frac{\partial^2 u}{\partial t^2} - a^2\nabla^2 u = 0$$

$$a = \sqrt{E/\rho}$$

§7.2 定解问题

$\boxed{\text{泛性条件}}$ + $\boxed{\text{定解条件}}$ = $\boxed{\text{定解问题}}$

共性　　个性

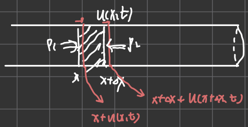
定解问题：
- 存在
- 唯一
- 稳定

个性：
① 边界条件 — 环境
② 初始条件 — 历史

边界条件：
- 自然
- 周期
- 阶跃 → 衔接
（可简化，可缺失若干点）

1) 波动问题.

ep 声音的传播.



粒子运动 ⟶ ρ 改变 ⟶ P 改变

压强 $p = f(\rho)$ $\quad P_0 = f(\rho_0)$

$1\ bar = 10^5 N/m$

$I = 20\log_{10}\left(\frac{\Delta p}{p_{ref}}\right)$

$P_{ref} = 2\times 10^{-10}\ bar$

$$\begin{cases} p = f(\rho) = p_0 + \Delta p \\ p_0 = f(\rho_0) \end{cases}$$

$$\underbrace{f(\rho_0+\Delta\rho)}_{p} = \underbrace{f(\rho_0)}_{p_0} + \underbrace{f'(\rho_0)\Delta\rho}_{\Delta p}$$

$$p_1 S - p_2 S = \rho_0 \cdot S \cdot \Delta x \cdot \frac{\partial^2}{\partial t^2} u(x,t)$$

$$\Delta p_1 - \Delta p_2 = \rho_0 \Delta x\, u_{tt}(x,t)$$

$$f'(\rho_0)\Delta\rho\Big|_{x=x} - f'(\rho_0)\Delta\rho\Big|_{x=x+\Delta x} = \rho_0 \Delta x\, U_{tt}(x,t) \quad ①$$

$$\rho_0 \Delta x S = \rho(\Delta x + U(x+\Delta x,t) - U(x,t))S$$

$$\Rightarrow \quad \rho_0 = \rho + \rho\frac{U(x+\Delta x,t) - U(x,t)}{\Delta x} = \rho + \rho\frac{\partial u}{\partial x}(x,t)$$

$$\Rightarrow \quad \Delta\rho\Big|_{x=x} = -\rho\frac{\partial u}{\partial x}(x,t) = -\rho_0\frac{\partial u}{\partial x}(x,t)$$

$$\Delta\rho\Big|_{x=x+\Delta x} = -\rho\frac{\partial u}{\partial x}(x+\Delta x,t) = -\rho_0\frac{\partial u}{\partial x}(x+\Delta x,t)$$

$$① \Rightarrow f'(\rho_0)\rho_0\left[\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right] = \rho_0 \Delta x\, U_{tt}(x,t)$$

$$\Rightarrow \quad f'(\rho_0)\frac{\frac{\partial u}{\partial x}\Big|_{x=x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x=x}}{\Delta x} = U_{tt}(x,t)$$

$$\frac{\partial^2 u}{\partial t^2} - f'(\rho_0)\cdot\frac{\partial^2 u}{\partial x^2} = 0$$

$$f'(\rho_0) = \frac{\partial p}{\partial \rho}\Big|_{\rho=\rho_0} = K = C_s^2$$

$$\Downarrow$$

$$\frac{\partial^2 u}{\partial t^2} - K\frac{\partial^2 u}{\partial x^2} = 0$$

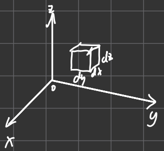
2) 输运问题

扩散：$\vec{\Gamma} = -D \cdot \nabla u$　　扩散定律　　$u_t - D\Delta u = f(\vec{r}, t)$

D：扩散系数

eg. 中子 (Fission)

$n +$ 铀 $\rightarrow$ 碎块 $+ \beta n$



$$dN = \Gamma|_x\, dy\,dz\,dt + \Gamma|_y\, dx\,dz\,dt + \Gamma|_z\, dx\,dy\,dt - \left(\Gamma|_{x+dx}\, dy\,dz\,dt + \Gamma|_{y+dy}\, dx\,dz\,dt + \Gamma|_{z+dz}\, dx\,dy\,dt\right) + \beta u\, dx\,dy\,dz\,dt$$

$$= \frac{\partial u}{\partial t} dx\,dy\,dz\,dt$$

$$\Rightarrow \frac{\partial u}{\partial t} = -\frac{\partial \Gamma}{\partial x} - \frac{\partial \Gamma}{\partial y} - \frac{\partial \Gamma}{\partial z} + \beta u = D\frac{\partial^2 u}{\partial x^2} + D\frac{\partial^2 u}{\partial y^2} + D\frac{\partial^2 u}{\partial z^2} + \beta u = \beta u + D\Delta u$$

$$\boxed{\frac{\partial u}{\partial t} - D\Delta u = \beta u}$$

Fission $\beta > 0$

uniform $\beta = 0$　　No origin / sink

attenuation / decrease / 衰减 $\beta < 0$

or

$$\boxed{\begin{aligned}&\frac{\partial u}{\partial t} - \nabla(D\nabla u) = \beta u\\ \Rightarrow\ &\frac{\partial u}{\partial t} - (\nabla D)(\nabla u) - D\Delta u = \beta u\end{aligned}}$$

3) 稳定场分布.

$$\Delta u = -\frac{\beta u}{D}$$

输运问题. 临界状态 $u_t = 0$

$\left.\begin{aligned}&\Delta u = -f(x,y,z,t)\\ &\Delta u = f(x,y,z)\end{aligned}\right\}$ 有源/汇

$\Delta u = 0$　　无源/汇

§7.2 定解条件.

一、初始条件

1) 波动

$u|_{t=0} = \varphi(x)$　初始位移

$u_t|_{t=0} = \psi(x)$　初始速度.

2) 输运

$u|_{t=0} = \varphi(\vec{r})$　初始分布.

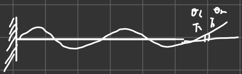
3) 无初始条件问题

如稳定场条件、受迫振动

二、边界条件

① 波动

1) 第一类边界条件



$u|_{x=0} = 0$ 固定边界条件

$u|_{x=l} = f(t)$ 自由边界条件.

中间段 $\begin{cases} T_1\cos\theta_1 - T_2\cos\theta_2 = 0 \\ T_1\sin\theta_1 - T_2\sin\theta_2 = \rho\Delta x u_{tt} \end{cases}$

$\frac{\partial u}{\partial x} = tg\theta$

边界 $\begin{cases} T_1\cos\theta_1 = T_2\cos\theta_2 = T_l \\ T_l\frac{\partial u}{\partial x} = \rho\Delta x u_{tt} \Rightarrow T_l \to 0 \end{cases}$

$\Downarrow$

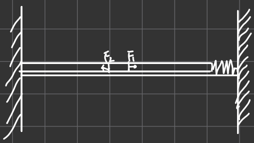
$T\frac{\partial^2 u}{\partial x^2} = \rho u_{tt}$ 即 $u_{tt} - \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = 0$

$u_x|_{x=a} = 0$.

2) 第二类边界条件

$u_x|_{x=l} = f(t)$

3) 第三类边界条件



$(u_x + Hu)|_{x=b} = f(t)$

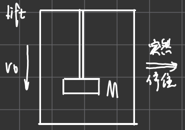
$u_{tt} - a^2 u_{xx} = 0$

$u|_{x=0} = 0$

$F_1|_{x+\Delta x} - F_2|_{x} = \rho\cdot\Delta x S\cdot u_{tt}$

$\frac{\Delta u}{\Delta x} = \frac{1}{Y}\frac{F}{S} = \frac{1}{Y}\cdot\sigma$

4) 其他类边界条件



$\xrightarrow[\text{停住}]{\text{突然}}$ 杆纵振动.

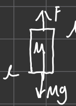
$$u_{tt}-\frac{Y}{\rho}u_{xx}=0$$

$$\frac{Mg}{S}=Y\frac{u|_{x=l}}{l}$$

$$\left(u|_{x=l}=\frac{Mg}{SY}l\right)$$

初始条件 $\begin{cases} u|_{t=0}=kx=\frac{Mg}{SY}x \\ u_t|_{t=0}=v_0 \end{cases}$

边界条件 $\begin{cases} u|_{x=0}=0 \\ u_x|_{x=l}=\frac{Mg}{YS}-\frac{M}{YS}u_{tt}|_{x=l} \end{cases}$



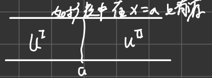
$$Mg-F=Mu_{tt}|_{x=l}$$

$$F=Mg-Mu_{tt}|_{x=l}=YSu_x|_{x=l}$$

注意事项:

①边界条件条件是整个时间范围内的边界条件 (某一时刻可忽略不计)

②外力与边界效应若为同源，不应重复考虑.



$$u_t-Du_{xx}=q\,\underbrace{\delta(x-a)}_{\text{量纲}\frac{1}{x}}$$

or

$$\begin{cases} u_t^I-Du_{xx}^I=0 \\ u_t^{II}-Du_{xx}^{II}=0 \end{cases}$$

5) 衔接条件

$$\begin{cases} u^I|_{x=a^-}=u^{II}|_{x=a^+} \\ Du_x^I|_{x=a}=q/2 \quad Du_x^{II}|_{x=a}=-q/2 \end{cases}$$

③注意量纲问题

④无边界问题.
$(x\to\infty)$

自然边界条件 $u|_{x\to\infty}$ 有限

上节导出的泛定方程，除杆的横振动方程以外，每处边界上只要一个边界条件. 以弦振动为例，就 x 这个自变数而言，弦振动方程中出现二阶导数 u_{xx}，是二阶微分方程，总共要求两个边界条件，即两端点各一个边界条件. 杆的横振动方程中出现四阶偏导数 u_{xxxx}，所以每个端点就要两个边界条件. 例如，端点 $x=a$ 被固定(图 7-11a)，那么这端点的位移始终为零，而且杆在该处的斜率亦为零，即

$$u|_{x=a}=0,\quad u_x|_{x=a}=0.\quad (\text{固定端})$$

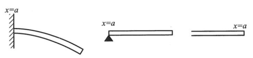
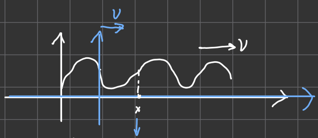


图 7-11

§7.3 行波法求解无限空间中的波动问题.

$$\begin{cases} u_{tt}-a^2u_{xx}=0, & -\infty<x<\infty \\ u|_{t=0}=\varphi(x) \\ u_t|_{t=0}=\psi(x) \end{cases}$$



$$u(x,t)=u(x-vt)=u(\xi)$$

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{du}{d\xi}\frac{\partial\xi}{\partial t}=u'(\xi)(-v) \\ \frac{\partial u}{\partial x}=\frac{du}{d\xi}\frac{\partial\xi}{\partial x}=u'(\xi) \end{cases}$$

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}[u'(\xi)(-v)]=v^2u''(\xi) \\ \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}[u'(\xi)]=u''(\xi) \end{cases}$$

$$\begin{cases} v^2\frac{d^2u}{d\xi^2}-a^2\frac{d^2u}{d\xi^2}=0 \\ u(x)=\varphi(x)=f(x)+g(x) \\ -vu'(x)=\psi(x)=-af'(x)+ag'(x) \end{cases}$$

$$\begin{cases} v^2=a^2\Rightarrow v=\pm a,\ u\text{任意} \\ \frac{d^2u}{d\xi^2}=0\Rightarrow u=A\xi+B\ \xrightarrow[u\text{有限}]{\xi\to\infty}\ A=0,\ u=B\ (\text{舍}) \end{cases}$$

$$u(x,t)=f(x-at)+g(x+at)$$

$$\begin{cases} u(x) = \varphi(x) = f(x) + g(x) \\ -\gamma u(x) = \psi(x) = -af'(x) + ag'(x) \end{cases} \Rightarrow \begin{cases} f(x) + g(x) = \varphi(x) \\ -f(x) + g(x) = \frac{1}{a}\int_{-\infty}^{x}\psi(\xi)d\xi + C \end{cases} \Rightarrow \begin{cases} f(x) = \frac{1}{2}\left[\varphi(x) - \frac{1}{a}\int_{-\infty}^{x}\psi(\xi)d\xi - C\right] \\ g(x) = \frac{1}{2}\left[\varphi(x) + \frac{1}{a}\int_{-\infty}^{x}\psi(\xi)d\xi + C\right] \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[\varphi(x-at) + \varphi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

达朗贝尔公式

$$\left(\frac{1}{a^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u(x,t) = 0$$

$$\left(\frac{1}{a}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{1}{a}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u(x,t) = 0$$

$\xi = x - at \quad \eta = x + at$

$$\begin{cases} x = \frac{1}{2}(\eta + \xi) \\ t = \frac{1}{2a}(\eta - \xi) \end{cases} \Rightarrow \frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}u(\xi,\eta) = 0$$

$$\Rightarrow u(\xi,\eta) = f(\xi) + g(\eta)$$

$$= f(x-at) + g(x+at)$$

$$\frac{\partial}{\partial\xi} = \frac{\partial x}{\partial\xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\xi}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2a}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial\eta} = \frac{\partial x}{\partial\eta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\eta}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2a}\frac{\partial}{\partial t}$$

记 $\Psi(x) \equiv \int_{-\infty}^{x}\psi(\xi)d\xi$

$$u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\left[\Psi(x+at) - \Psi(x-at)\right]$$

行波解

$$u_{tt} - a^2 u_{xx} = \sin u$$

$$u(x,t) = u(\xi) = u(x - vt)$$

$$v^2 u_{\xi\xi} - a^2 u_{\xi\xi} = \sin u$$

$$\frac{d^2u}{d\xi^2} = \frac{\sin u}{v^2 - a^2}$$

$$\frac{du}{d\xi}\frac{d^2u}{d\xi^2} = \frac{\sin u}{v^2 - a^2}\frac{du}{d\xi}$$

$$\frac{1}{2}\frac{d}{d\xi}\left(\frac{du}{d\xi}\right)^2 = \frac{1}{v^2 - a^2}\frac{d}{d\xi}\cdot\cos u$$

$$\frac{du}{d\xi} = \sqrt{\frac{2\cos u}{v^2 - a^2} + C_1}$$

$$d\xi = \frac{du}{\sqrt{\frac{2\cos u}{v^2 - a^2} + C_1}}$$

半无限长

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 \le x < \infty \\ u|_{t=0} = \varphi(x) & u|_{x=0} = 0 \\ u_t|_{t=0} = \psi(x) \end{cases}$$

Def $\Phi(x) = \begin{cases} \varphi(x) & x > 0 \\ -\varphi(-x) & x < 0 \end{cases}$

$\Psi(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \end{cases}$

$\Downarrow$

$u(x,t) = -u(-x,t)$

则 $\begin{cases} u|_{x=0} \Rightarrow \text{奇延拓} \\ u_x|_{x=0} \Rightarrow \text{偶延拓.} \end{cases}$

若 $u|_{x=0} = h(t)$

则令 $u(x,t) = f(x)h(t) + \tilde{u}(x,t)$

令 $\begin{cases} f(0) = 1 \\ h''(t)f(x) - a^2 f''(x)h(t) = 0 \end{cases}$

例 $u|_{x=0} = A\sin\omega t$

猜想 $f(x) = \cos\beta x$

$-A\omega^2\sin\omega t\cos\beta x + a^2A\beta^2\cos\beta x\sin\omega t = 0$

$-\omega + a^2\beta^2 = 0 \quad \beta^2 = \frac{\omega^2}{a^2}$

则 $f(x) = \cos\frac{\omega}{a}x$

$u(x,t) = A\cos\frac{\omega}{a}x\sin\omega t + \tilde{u}(x,t)$

$$\begin{cases} \tilde{u}_{tt} - a^2\tilde{u}_{xx} = 0 \\ \tilde{u}|_{x=0} = 0 \\ \tilde{u}|_{t=0} = \varphi(x) \\ \tilde{u}_t|_{t=0} = \psi(x) - \omega A\cos\frac{\omega}{a}x \end{cases}$$

$\Downarrow$

$$\tilde{u}(x,t) = \frac{1}{2}\left[\varphi(x-at) + \varphi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\left(\psi(x) - A\omega\cos\frac{\omega}{a}x\right)dx$$

§8. 分离变数法. 求解定解问题

本质: Fourier 级数

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & 0 \le x \le l \\ u|_{x=0} = 0 \\ u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

$u(x,t) = X(x)T(t)$

$X(x)T''(t) - a^2 X''(x)T(t) = 0$

$$\begin{cases} X(x)T(0) = \varphi(x) \\ X(x)T'(0) = \psi(x) \end{cases}$$

$\Rightarrow \dfrac{T''(t)}{a^2 T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$ (只能为常值)

$$\begin{cases} X(0)T(t) = 0 \\ X(l)T(t) = 0 \end{cases} \Rightarrow \begin{cases} X(0) = 0 \\ X(l) = 0 \end{cases}$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ T''(t) + a^2\lambda T(t) = 0 \end{cases}$$

$\Rightarrow$ ① $\lambda > 0$

$X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$

$(B = 0, \quad \sqrt{\lambda}l = n\pi)$

$X(x) = A\sin\frac{n\pi}{l}x, \quad \lambda = \frac{(n\pi)^2}{l^2}$

$X_n(x) = A_n\sin\frac{n\pi}{l}x$, $\lambda = \frac{n^2\pi^2}{l^2}$ 为本征值.

本征函数 (本征解)

② $\lambda < 0$

$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$$\begin{cases} A + B = 0 \\ Ae^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} = 0 \end{cases} \Rightarrow$$ $A = B = 0 \Rightarrow X(x) = 0$ 不成立. 舍去.

③ $\lambda = 0$.

$X(x) = Ax + B$

$$\begin{cases} B = 0 \\ Al + B = 0 \end{cases} \Rightarrow$$ $A = B = 0 \Rightarrow X(x) = 0$ 不成立. 舍去.

解: $T''(t) + a^2\lambda T(t) = 0 \Leftarrow$

得: $T_n(t) = C_n\sin\frac{an\pi}{l}t + D_n\cos\frac{an\pi}{l}t$

$\Downarrow$

$u_n(x,t) = \left(\tilde{C}_n\sin\frac{an\pi}{l}t + \tilde{D}_n\cos\frac{an\pi}{l}t\right)\cdot\sin\frac{n\pi}{l}x$ 本征振动

驻波

$(n = 1, 2, \cdots)$

$x = \frac{kl}{n}$ $(k = 0, 1, 2, \cdots, n) \Rightarrow \sin\frac{n\pi}{l}x = 0$

驻点 (共 $n+1$ 个)

半波长 $\frac{l}{n}$、波长 $\frac{2l}{n}$

$u(x,t) = \sum_{n=1}^{\infty}\left[\tilde{C}_n\sin\frac{an\pi}{l}t + \tilde{D}_n\cos\frac{an\pi}{l}t\right]\sin\frac{n\pi}{l}x$

代入初始条件 $\begin{cases} u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$

$\tilde{D}_n\int_0^l \frac{1-\cos\frac{2n\pi}{l}x}{2}dx = \frac{l}{2}\tilde{D}_n$

$\Uparrow$

要求 $\tilde{C}_n$、$\tilde{D}_n$ 使级数收敛

$$\begin{cases} \sum_{n=1}^{\infty}\tilde{D}_n\sin\frac{n\pi}{l}x = \varphi(x) \xRightarrow[\text{积分}]{\text{乘}\sin\frac{m\pi}{l}x} \sum_{n=1}^{\infty}\underbrace{\tilde{D}_n\int_0^l\sin\frac{n\pi}{l}x\cdot\sin\frac{m\pi}{l}x\,dx}_{} = \int_0^l\varphi(x)\sin\frac{m\pi}{l}x\,dx \\ \sum_{n=1}^{\infty}\tilde{C}_n\frac{an\pi}{l}\sin\frac{n\pi}{l}x = \psi(x) \xRightarrow[\text{积分}]{\text{乘}\sin\frac{m\pi}{l}x} \sum_{n=1}^{\infty}\underbrace{\tilde{C}_n\frac{an\pi}{l}\int_0^l\sin\frac{n\pi}{l}x\sin\frac{m\pi}{l}x\,dx}_{} = \int_0^l\psi(x)\sin\frac{m\pi}{l}x\,dx \end{cases}$$

$\Rightarrow \boxed{\tilde{D}_n = \frac{2}{l}\int_0^l\varphi(x)\sin\frac{m\pi}{l}x\,dx}$

$\Rightarrow \boxed{\tilde{C}_n = \frac{2}{an\pi}\int_0^l\psi(x)\sin\frac{m\pi}{l}x\,dx}$

$\| \quad \tilde{C}_n\frac{an\pi}{2}$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left[\tilde{C}_n\sin\frac{an\pi}{l}t + \tilde{D}_n\cos\frac{an\pi}{l}t\right]\sin\frac{n\pi}{l}x$,
$$\begin{cases} \tilde{C}_n = \frac{2}{an\pi}\int_0^l\psi(x)\sin\frac{n\pi}{l}x\,dx \\ \tilde{D}_n = \frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx \end{cases}$$

n次谐波最大振幅逐渐衰减

ii) $u_{tt} - a^2 u_{xx} = f(x,t)$

$\Rightarrow$ 通解 + 特解

iii) $\begin{cases} u_x|_{x=0} = 0 \\ u_x|_{x=l} = 0 \end{cases}$ （第二类边界条件）

$\Rightarrow$ 通解为 cos(...) 的形式 $u(x,t) = \sum_{n=1}^{\infty} T_n(t) \cos \frac{n\pi}{l} x$

iv) $\begin{cases} u|_{x=0} = 0 \\ u_x|_{x=l} = 0 \end{cases}$ （一、二类边界条件） $\begin{cases} u|_{x=l} = 0 \\ u_x|_{x=0} = 0 \end{cases}$

$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} T_n(t) \sin \lambda_n x$ $\Rightarrow u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos \lambda_n x$

$\lambda_n = \frac{(2n+1)\pi}{2l}$ $\lambda_n = \frac{(2n+1)\pi}{2l}$

$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} T_n(t) \sin \frac{(2n+1)\pi}{2l} x$ $u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos \frac{(2n+1)\pi}{2l} x$

v) $\begin{cases} u|_{x=0} = 0 \\ (u + H u_x)|_{x=l} = 0 \end{cases}$ 第三类边界条件

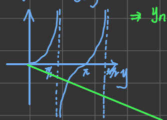
$\Rightarrow \begin{cases} X(0) = 0 \\ X(l) + H X'(l) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases} \Rightarrow \begin{cases} B = 0 \\ A \sin\sqrt{\lambda} l + H A \sqrt{\lambda} \cos\sqrt{\lambda} l = 0 \end{cases}$

$\sqrt{H^2+1} \sin(\sqrt{\lambda} l + \varphi) = 0$

$\sqrt{\lambda} l + \varphi = n\pi$

$\lambda_n = \left(\frac{n\pi - \varphi}{l}\right)^2$

$\tg y = -\frac{H}{l} y$ （$y = \sqrt{\lambda} l$） $\Rightarrow y_n,\ \lambda_n = \frac{y_n^2}{l^2}$



$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{y_n}{l} x$

or

$\sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi - \varphi}{l} x$

非正交.

§8.2. 傅里叶级数法求解边界齐次的非齐次方程.

方法论：1) 先求对应齐次方程.

2) 再满足齐次边界条件 本征值问题. $\Rightarrow \lambda_n,\ X_n(x)$

3) 令 $u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$, $f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$

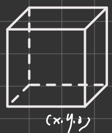
$u_{tt} - a^2 u_{xx} = f(x,t)$

$\Rightarrow T_n''(t) - a^2 \lambda_n T_n(t) = f_n(t)$

$n + U^{235} \longrightarrow \beta n + 碎块 + h\nu$

$u(x,y,z,t)$

单位体积中子数



$u_t - D\Delta u = \beta u$ （中子扩散）

$$\begin{cases} u|_{x=0} = 0,\ u|_{x=a} = 0 \\ u|_{y=0} = 0,\ u|_{y=a} = 0 \\ u|_{z=0} = 0,\ u|_{z=a} = 0 \end{cases}$$

临界体积

$u_t = 0$

$\Downarrow$ 与t无关

$$\Delta u + \frac{\beta}{D} u = 0$$

$$u(x,y,z) = X(x)Y(y)Z(z)$$

$$X''(x) Y Z + X Y''(y) Z + X Y Z''(z) + \frac{\beta}{D} X Y Z = 0$$

$$\Downarrow$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + \frac{\beta}{D} = 0$$

$$\Downarrow$$

$$-\lambda_x - \lambda_y - \lambda_z + \frac{\beta}{D} = 0 \qquad \begin{cases} X''(x) + \lambda_x X(x) = 0 \\ X(0) = X(a) = 0 \\ Y''(y) + \lambda_y Y(y) = 0 \\ Y(0) = Y(a) = 0 \\ Z''(z) + \lambda_z Z(z) = 0 \\ Z(0) = Z(a) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} X = A_x \sin\sqrt{\lambda_x}\, x + B_x \cos\sqrt{\lambda_x}\, x \\ Y = A_y \sin\sqrt{\lambda_y}\, y + B_y \cos\sqrt{\lambda_y}\, y \\ Z = A_z \sin\sqrt{\lambda_z}\, z + B_z \cos\sqrt{\lambda_z}\, z \end{cases} \quad \begin{cases} B_x = 0 \\ B_y = 0 \\ B_z = 0 \end{cases} \quad \begin{cases} \lambda_x = \frac{n^2\pi^2}{a^2} \\ \lambda_y = \frac{m^2\pi^2}{a^2} \\ \lambda_z = \frac{k^2\pi^2}{a^2} \end{cases}$$

$$\frac{\pi^2}{a^2}(m^2 + n^2 + k^2) = \frac{\beta}{D}$$

$$a^2 = \frac{D}{\beta}\pi^2 (m^2 + n^2 + k^2)$$

临界体积 $V = a_{min}^3$ $\left(a_{min} = \sqrt{\frac{3D}{\beta}} \cdot \pi\right)$

$$= \left(\frac{3D}{\beta}\pi^2\right)^{3/2}$$

§8.3 非齐次边界的处理

$$\begin{cases} u_{tt}-a^2u_{xx}=f(x,t) \\ u|_{t=0}=\varphi(x) \\ u_t|_{t=0}=\psi(x) \\ u|_{x=0}=\mu(t) \\ u|_{x=l}=\nu(t) \end{cases}$$

$$\tilde{u}(x,t)=u(x,t)-\mu(t)+\frac{x}{l}[\mu(t)-\nu(t)]$$

$$\begin{cases} \tilde{u}|_{x=0}=u|_{x=0}-\mu(t)=0 \\ \tilde{u}|_{x=l}=u|_{x=l}-\nu(t)=0 \end{cases}$$

$$\Downarrow$$

$$\tilde{u}_{tt}-\mu''(t)+\frac{x}{l}[\mu''(t)-\nu''(t)]-a^2\tilde{u}_{xx}$$

$$\begin{cases} \tilde{u}_{tt}-a^2\tilde{u}_{xx}=f(x,t)+\mu''(t)-\frac{x}{l}[\mu''(t)-\nu''(t)]=\tilde{f}(x,t) \\ \tilde{u}|_{t=0}=\varphi(x)-\mu(0)+\frac{x}{l}[\mu(0)-\nu(0)]=\tilde{\varphi}(x) \\ \tilde{u}_t|_{t=0}=\psi(x)-\mu'(0)+\frac{x}{l}[\mu'(0)-\nu'(0)]=\tilde{\psi}(x) \end{cases}$$

if $\begin{cases} u_x|_{x=0}=\mu(t) \\ u_x|_{x=l}=\nu(t) \end{cases}$

$$\tilde{u}(x,t)=u(x,t)-x\mu(t)+\frac{x^2}{2l}[\mu(t)-\nu(t)]$$

if $\begin{cases} u|_{x=0}=\mu(t) \\ u_x|_{x=l}=\nu(t) \end{cases}$

$$\hat{u}(x,t)=u(x,t)-\mu(t)+\frac{x^2}{2l}\nu(t)$$

if $\begin{cases} u_x|_{x=0}=\mu(t) \\ u|_{x=l}=\nu(t) \end{cases}$

$$\hat{u}(x,t)=u(x,t)-x\mu(t)+\frac{x}{l}(\mu(t)-\nu(t))$$

Epx：输运问题 + 第三类边界条件

$$u_t-a^2u_{xx}=0, \quad a^2=\frac{k}{\rho c}$$

$$\begin{cases} u|_{x=0}=u_0 \\ \left(\frac{\partial u}{\partial x}+\frac{h}{k}u\right)\Big|_{x=l}=\frac{h}{k}\theta \\ u|_{t=0}=\varphi(x) \end{cases}$$

$$-k\frac{\partial u}{\partial x}\Big|_{x=l-\Delta x}-h(u|_{x=l}-\theta)=\Delta u\,\rho\,\Delta x\,c \overset{\Delta x\to 0}{\Rightarrow} 0$$

令 $u=\hat{u}+A_0+B_0x$

$$u|_{x=0}=\hat{u}|_{x=0}+A_0=u_0 \Rightarrow A_0=u_0$$

$$\left(\frac{\partial u}{\partial x}+Hu\right)\Big|_{x=l}=\left(\frac{\partial\hat{u}}{\partial x}+B_0+H\hat{u}+Hu_0+HB_0x\right)\Big|_{x=l}$$

$$=\frac{\partial\hat{u}}{\partial x}\Big|_{x=l}+H\hat{u}|_{x=l}+B_0+HB_0l+Hu_0$$

$$=H\theta$$

$$\Rightarrow B_0=\frac{H\theta-Hu_0}{1+Hl}$$

$$\Rightarrow u=\hat{u}+u_0+\frac{H(\theta-u_0)}{1+Hl}x$$

$$\Downarrow$$

$$\hat{u}_t-a^2\hat{u}_{xx}=0$$

$$\begin{cases} \hat{u}|_{t=0}=\varphi(x)-u_0+\frac{H(u_0-\theta)}{1+Hl}x \\ \hat{u}|_{x=0}=0 \\ \left(\frac{\partial\hat{u}}{\partial x}+H\hat{u}\right)\Big|_{x=l}=0 \end{cases}$$

求解：$\hat{u}(x,t)=X(x)T(t)$

$$\begin{cases} X''+\lambda X=0 \Rightarrow X_n=A_n\sin\sqrt{\lambda}x+B_n\cos\sqrt{\lambda}x \\ T'+a^2\lambda T=0 \Rightarrow T_n=C_ne^{-a^2\lambda t} \end{cases}$$

代入边界条件：$\begin{cases} B_n=0 \\ A_n\sqrt{\lambda}\cos\sqrt{\lambda}l+HA_n\sin\sqrt{\lambda}l=0 \end{cases}$

$$\tan\sqrt{\lambda}l=-\frac{\sqrt{\lambda}}{H}$$

$$\mathrm{tg}\,y=-\frac{y}{Hl}, \quad \lambda_n=\frac{y_n^2}{l^2}$$

$$\hat{u}_n(x,t)=D_ne^{-a^2\lambda t}\cdot\sin\sqrt{\lambda}x$$

$\boxed{\text{验证正交性}}$

代入初始条件

$$D_n=\frac{1}{\left(\frac{l}{2}+\frac{l\sin 2x_n}{4x_n}\right)}\int_0^l\left[\varphi(x)-u_0+\frac{H(u_0-\theta)}{1+Hl}\right]\sin\sqrt{\lambda}x\,dx$$

$$\hat{u}_n(x,t)=\frac{1}{\left(\frac{l}{2}+\frac{l\sin 2x_n}{4x_n}\right)}\int_0^l\left[\varphi(x)-u_0+\frac{H(u_0-\theta)}{1+Hl}\right]\sin\frac{y_n}{l}x\,dx\,e^{-a^2\frac{y_n^2}{l^2}t}\cdot\sin\frac{y_n}{l}x$$

$$\hat{u}(x,t)=\sum_{n=0}^{\infty}\frac{1}{\left(\frac{l}{2}+\frac{l\sin 2x_n}{4x_n}\right)}\int_0^l\left[\varphi(x)-u_0+\frac{H(u_0-\theta)}{1+Hl}\right]\sin\frac{y_n}{l}x\,dx\cdot e^{-a^2\frac{y_n^2}{l^2}t}\sin\frac{y_n}{l}x+u_0+\frac{H(\theta-u_0)}{1+Hl}x$$

一般问题，

$$u_{tt} - a^2 u_{xx} = f(x,t)$$

$$\begin{cases} u|_{x=0} = \mu(t) \\ u|_{x=l} = \nu(t) \\ u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

令 $u(x,t) = \tilde{u}(x,t) + \mu(t) + \frac{x}{l}[\nu(t) - \mu(t)]$

$$\tilde{u}_{tt} - a^2 \tilde{u}_{xx} = f(x,t)$$

$$\begin{cases} \tilde{u}|_{x=0} = 0 \\ \tilde{u}|_{x=l} = 0 \\ \tilde{u}|_{t=0} = \bar{\varphi}(x) \\ \tilde{u}_t|_{t=0} = \bar{\psi}(x) \end{cases}$$

$\tilde{u} = XT$ $\begin{cases} X'' + \lambda X = 0 \\ T'' + a^2 \lambda T = 0 \end{cases}$

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi}{l} x$$

$$\sum_{n=1}^{\infty} \left[T_n''(t) + a^2 \frac{n^2\pi^2}{l} T_n(t) \right] \sin\frac{n\pi}{l} x = f(x,t)$$

$$\left[T_n''(t) + a^2 \frac{n^2\pi^2}{l} T_n(t) \right] = \frac{2}{l} \int_0^l f(x,t) \sin\frac{n\pi}{l} x \, dx = \bar{F}(t)$$

解 $py'' + qy = Q(x)$

齐次通解：y_1, y_2

求特解 $y = c_1 y_1 + c_2 y_2$

$y' = c_1 y_1' + c_2 y_2'$

$(c_1' y_1 + c_2' y_2 = 0)$

$$\Rightarrow \begin{cases} c_1 = \int \frac{Q(x)}{y_1' - y_2' \frac{y_1}{y_2}} dx \\ c_2 = \int \frac{Q(x)}{y_2' - y_1' \frac{y_2}{y_1}} dx \end{cases} \Longrightarrow \text{通解 } y = Ay_1 + By_2 + c_1(x) y_1 + c_2(x) y_2$$

冲量定理。
引入

$$u(x,t) \equiv \int_0^t v(x,t;\tau) d\tau \qquad f(x,t) = \int_0^{t^+} f(x,\tau) \delta(t-\tau) d\tau$$

$$\frac{\partial u}{\partial t} = \int_0^t v_t(x,t;\tau) d\tau + \underset{0}{\underbrace{v(x,t;t)}_{\text{def}}}$$

$$\frac{\partial^2 u}{\partial t^2} = \int_0^t v_{tt}(x,t;\tau) d\tau + \underset{0}{\underbrace{v_t(x,t;t)}_{\text{def}}} \Longrightarrow \int_0^t v_{tt}(x,t;\tau) d\tau - a^2 \int_0^t v_{xx}(x,t;\tau) d\tau$$

$$= \int_0^t f(x,\tau) \delta(t-\tau) d\tau$$

$$\int_0^t v_{tt}(x,t;\tau)\,d\tau - a^2\int_0^t v_{xx}(x,t;\tau)\,d\tau = \int_0^t f(x,\tau)\,\delta(t-\tau)\,d\tau$$

$$\Rightarrow \begin{cases} u|_{t=0}=0 \\ u_t|_{t=0}=0 \end{cases} \text{auto}$$

$$\begin{cases} u|_{x=0}=0 \\ u|_{x=l}=0 \end{cases}$$

$$v_{tt}(x,t;\tau) - a^2 v_{xx}(x,t;\tau) = f(x,\tau)\,\delta(t-\tau)$$

当 $t<\tau$ 时 $v=0$, $v_t=0$

当 $t=\tau^+$ 时

$$\int_{\tau^-}^{\tau^+} v_{tt}(x,t;\tau)\,dt - a^2\int_{\tau^-}^{\tau^+} v_{xx}(x,t;\tau)\,dt = \int_{\tau^-}^{\tau^+} f(x,\tau)\,\delta(t-\tau)\,dt = f(x,\tau)$$

$$\int_{\tau^-}^{\tau^+} v_{tt}\,dt = v_t(x,t;\tau)\Big|_{t=\tau^-}^{t=\tau^+} = v_t(x,\tau^+;\tau) - \underbrace{v_t(x,\tau^-;\tau)}_{=0}, \qquad \int_{\tau^-}^{\tau^+} v_{xx}\,dt = 0$$

$$\Downarrow$$

$$v_t(x,\tau^+;\tau) = f(x,\tau)$$

$$\Downarrow$$

$$\begin{cases} v_{tt}(x,t;\tau) - a^2 v_{xx}(x,t;\tau) = 0 \\ v(x,\tau^+;\tau)|_{x=0}=0 \\ v(x,\tau^+;\tau)|_{x=l}=0 \\ v(x,t;\tau)|_{t=\tau^+}=0 \\ v_t(x,t;\tau)|_{t=\tau^+} = f(x,\tau) \end{cases} \xrightarrow{\tilde{t}=t-\tau^+,\ (t-\tau)} \quad v_{\tilde{t}\tilde{t}} - a^2 v_{xx} = 0, \quad \begin{cases} v|_{x=0}=0 \\ v|_{x=l}=0 \\ v|_{\tilde{t}=0}=0 \\ v_{\tilde{t}}|_{\tilde{t}=0} = f(x,\tau) \end{cases}$$

§9 特殊常微分方程的级数解及本征值问题

§9.1 方程导出

球坐标 $\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e}_\varphi$ $\qquad \frac{\partial\vec{e}_r}{\partial r}=\frac{\partial\vec{e}_\theta}{\partial r}=\frac{\partial\vec{e}_\varphi}{\partial r}=0$

$\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta$ $\qquad \frac{\partial\vec{e}_r}{\partial\theta}=\vec{e}_\theta,\quad \frac{\partial\vec{e}_\theta}{\partial\theta}=-\vec{e}_r\quad \frac{\partial\vec{e}_\varphi}{\partial\theta}=\frac{\partial(\vec{e}_r\times\vec{e}_\theta)}{\partial\theta}=\frac{\partial\vec{e}_r}{\partial\theta}\times\vec{e}_\theta+\frac{\partial\vec{e}_\theta}{\partial\theta}\times\vec{e}_r=0$

$\frac{\partial}{\partial r}\cdot\vec{e}_r$ $\qquad \frac{\partial\vec{e}_r}{\partial\varphi}=\vec{e}_\varphi,\quad \frac{\partial\vec{e}_\theta}{\partial\varphi}=\vec{e}_\varphi\quad \frac{\partial\vec{e}_\varphi}{\partial\varphi}=-\vec{e}_r$

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$\vec{e}_r(\theta,\varphi)\qquad \vec{e}_\theta(r,\varphi)\qquad \vec{e}_\varphi(r)$

$$\Delta:\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Delta u=0$$

$$u(r,\theta,\varphi)=R(r)Y(\theta,\varphi)$$

$$\Rightarrow Y(\theta,\varphi)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right)+\frac{R(r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right)+\frac{R(r)}{r^2\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2}=0$$

$$-\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)=\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta}+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}=-\lambda$$

$$r^2R''+2rR'-\lambda R=0$$

$$\sin^2\theta\, Y''\big|_\theta+Y''\big|_\varphi+\sin\theta\cos\theta\, Y'\big|_\theta+\lambda\sin^2\theta\, Y=0$$

$$Y=H(\theta)\Phi(\varphi)$$

$$s^2\frac{H''}{H}+\frac{scH'}{H}+\lambda s^2=-\frac{\Phi''}{\Phi}=m^2$$

$$\begin{cases} r^2R''+2rR'-\lambda R=0 \xRightarrow{\lambda=l(l+1)} R(r)=C_l r^l+D_l\frac{1}{r^{l+1}} \\ \Phi''+m^2\Phi=0 \Longrightarrow \Phi(\varphi)=A_m\cos m\varphi+B_m\sin m\varphi \\ \sin^2\theta H''+\sin\theta\cos\theta H'+(\lambda\sin^2\theta-m^2)H=0 \end{cases}$$

勒让德方程 $(m=0)$

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)+\lambda\sin^2\theta H=0$$

柱坐标：

$$\Delta u=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=0$$

$$u(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$$

$$\frac{1}{\rho}\Phi Z\frac{\partial}{\partial\rho}\rho\cdot R'+\frac{1}{\rho^2}RZ\cdot\Phi''+R\Phi Z''=0$$

$$\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho R'+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}=-\frac{Z''}{Z}=-\mu$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\rho R'+\mu\rho^2=-\frac{\Phi''}{\Phi}=\lambda$$

$$\Rightarrow \begin{cases} Z'' - \mu Z = 0 \\ \Phi'' + \lambda\Phi = 0 \\ \rho^2 R'' + \rho R' + (\mu\rho^2 - \lambda)R = 0 \end{cases}$$

$\mu > 0$，m阶 Bessel 方程.

$\mu < 0$，虚宗量Bessel方程.

波动.

$$u_{tt} - a^2 \Delta u = 0$$

$$u = V(\vec{r})\,T(t)$$

$$V(\vec{r})\,T''(t) - a^2 T(t)\,\Delta V(\vec{r}) = 0$$

$$\frac{T''(t)}{a^2 T(t)} = \frac{\Delta V(\vec{r})}{V(\vec{r})} = -\lambda$$

$$\Delta V(\vec{r}) + \lambda V(\vec{r}) = 0$$

亥姆霍兹方程（波动方程）

输运

$$u_t - a^2 \Delta u = 0$$

$$u = V(\vec{r})\,T(t)$$

$$V(\vec{r})\,T'(t) - a^2 T(t)\,\Delta V(\vec{r}) = 0$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{\Delta V(\vec{r})}{V(\vec{r})} = -\lambda$$

$$\Delta V(\vec{r}) + \lambda V(\vec{r}) = 0$$

亥姆霍兹方程

亥姆霍兹方程

1) 球坐标下，分离变量.

$$\Delta V(\vec{r}) + k^2 V(\vec{r}) = 0$$

$$V(\vec{r}) = R(r)\,Y(\theta,\varphi)$$

$$\Rightarrow \begin{cases} r^2 R'' + 2rR' - l(l+1)R + k^2 r^2 R = 0 \\ \sin\theta \dfrac{\partial}{\partial\theta}\sin\theta\dfrac{\partial Y}{\partial\theta} + \dfrac{\partial^2 Y}{\partial\varphi^2} + l(l+1)Y\sin^2\theta = 0 \end{cases}$$

球 Bessel 方程

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]H(\theta) = 0$$

缔合勒让德方程

$m=0 \Rightarrow$ 勒让德方程

2) 柱坐标下

$$V(\vec{r}) = R(\rho)\,\Phi(\varphi)\,Z(z)$$

$$\begin{cases} Z''(z) - \mu Z(z) = 0 \\ \Phi''(\varphi) + m^2\Phi(\varphi) = 0 \\ \rho^2 R'' + \rho R' + \left[(k^2+\mu)\rho^2 - m^2\right]R = 0 \end{cases}$$

$k^2+\mu > 0$　Bessel方程.

$k^2+\mu < 0$　虚宗量 Bessel 方程

§9.2 常点邻域上级数解二阶常微分方程.

$$y''(x)+p(x)y'(x)+q(x)y(x)=0$$

$$\begin{cases} y'(x)\big|_{x=0}=C_1 \\ y(x)\big|_{x=0}=C_2 \end{cases}$$

$\Downarrow$ 复变.

$$\frac{d^2w(z)}{dz^2}+p(z)\frac{dw(z)}{dz}+q(z)w(z)=0$$

$p(z),q(z)$ 在 $z=z_0$ 邻域上解析

$w(z)$ 在 $z=z_0$ 邻域上存在解析函数.

$$w(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

$\lim\limits_{x\to x_0}p(x),q(x)$ 有限.

x_0 称为方程的常点. 泰勒级数.

若 $p(x)$ $q(x)$ 其中一发散 则 x_0 为奇点. $\begin{cases}\text{正则}:\text{洛朗级数}\\ \text{一般}\end{cases}$

$$y''(x)+m^2y(x)=0$$

$(p(x)=0,\ q(x)=m^2)$.

$(x=0$ 为方程常点$)$

$$y(x)\overset{\text{Taylor}}{=\!=}\sum_{n=0}^{\infty}a_nx^n$$

$$\sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}+m^2\sum_{n\geq 0}^{\infty}a_nx^n=0$$

令 $k=n-2$, $n=k+2$

有 $a_{k+2}(k+2)(k+1)x^k$

令 $k=n$

有 $m^2a_kx^k$

$\Rightarrow$ $\underline{[a_{k+2}(k+2)(k+1)+m^2a_k]x^k=0}$

$$a_{k+2}=-\frac{m^2}{(k+2)(k+1)}a_k$$

$$a_2=-\frac{m^2}{2\times1}a_0=-\frac{m^2}{2!}a_0$$

$$a_3=-\frac{m^2}{3\times2}a_1$$

$$a_4=-\frac{m^2}{4\times3}a_2=\frac{m^4}{4!}a_0$$

$$a_5=\frac{m^4}{5!}a_1$$

$$a_6=-\frac{m^6}{6!}a_0$$

$$\Rightarrow\begin{cases}a_{2k}=(-1)^k\dfrac{m^{2k}}{(2k)!}a_0\\ a_{2k+1}=(-1)^k\dfrac{m^{2k}}{(2k+1)!}a_1\end{cases}\Rightarrow y(x)=\sum_{k=0}^{\infty}a_{2k}x^{2k}+\sum_{k=0}^{\infty}a_{2k+1}x^{2k+1}$$

$$=a_0\sum_{k=0}^{\infty}(-1)^k\frac{m^{2k}}{2k!}x^{2k}+\frac{a_1}{m}\sum_{k=0}^{\infty}(-1)^k\frac{m^{2k+1}}{(2k+1)!}x^{2k+1}$$

$$=a_0\cos mx+\frac{a_1}{m}\sin mx$$

$$\overset{\text{def}}{=}a_0\cos mx+\tilde{a}_1\sin mx$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]H(\theta)=0$$

$x\equiv\cos\theta$

$dx=-\sin\theta d\theta$

$\sin^2\theta=1-\cos^2\theta=1-x^2$

$(-1<x<1)$

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{\sin^2\theta}{-\sin\theta}\frac{dH}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]H=0$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\tilde{H}}{dx}\right)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\tilde{H}(x)=0$$

缔合勒让德方程

Legendre

$$(1-x^2)\tilde{H}''(x)-2x\tilde{H}'(x)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\tilde{H}(x)=0$$

$$\begin{cases} p(x)=\dfrac{-2x}{1-x^2} \\ q(x)=\dfrac{l(l+1)}{1-x^2}-\dfrac{m^2}{(1-x^2)^2} \end{cases}$$

$x=0$ 为常点

$x=1,-1$ 为奇点

$$\tilde{H}(x)=\sum_{k=0}^{\infty}a_k x^k$$

$$\sum_{k=2}^{\infty}a_k k(k-1)x^{k-2}-\frac{2x}{1-x^2}\sum_{k=1}^{\infty}a_k\cdot kx^{k-1}+\left[\frac{l(l+1)}{1-x^2}-\frac{m^2}{(1-x^2)^2}\right]\sum_{k=0}^{\infty}a_kx^k=0$$

$$\sum_{k=0}^{\infty}a_{k+2}(k+2)(k+1)x^k-\frac{2x}{1-x^2}\sum_{k=0}^{\infty}a_{k+1}(k+1)x^k+\left[\frac{l(l+1)}{1-x^2}-\frac{m^2}{(1-x^2)^2}\right]\sum_{k=0}^{\infty}a_kx^k=0$$

$m=0$

$$\sum_{k=0}^{\infty}a_{k+2}(k+2)(k+1)x^k-\sum_{k=2}^{\infty}a_k\cdot k(k-1)x^k-\sum_{k=1}^{\infty}2a_k kx^k+l(l+1)\sum_{k=0}^{\infty}a_kx^k$$

$$a_{k+2}(k+2)(k+1)-a_kk(k-1)-2a_k\cdot k+l(l+1)a_k=0$$

$$\Rightarrow a_{k+2}=\frac{k(k+1)-l(l+1)}{(k+2)(k+1)}a_k=\frac{(k-l)(k+l+1)}{(k+2)(k+1)}a_k$$

收敛半径 $R=\lim_{k\to\infty}\frac{(k+2)(k+1)}{(k-l)(k+l+1)}=1$

$$\begin{cases} a_2=\dfrac{-l(l+1)}{2\times1}a_0 \\ a_3=\dfrac{(1-l)(2+l)}{3\times2}a_1 \\ a_4=\dfrac{(2-l)(3+l)}{4\times3}a_2=\dfrac{(2-l)(0-l)(3+l)(1+l)}{4!}a_0 \end{cases}$$

$$a_5=\frac{(3-l)(4+l)}{5\times4}a_3=\frac{(3-l)(1-l)(4+l)(2+l)}{5!}a_1$$

$$a_6=\frac{(4-l)(5+l)}{6\times5}a_4=\frac{(4-l)(2-l)(0-l)(5+l)(3+l)(1+l)}{6!}a_0$$

$P_l(x)$ Legendre 多项式

$$\begin{cases} a_{2k}=\dfrac{\prod_{i=1}^{k}(2i-2-l)\prod_{i=1}^{k}(2i-1+l)}{(2k)!}a_0=\dfrac{(2k+2[\frac{l}{2}]-l)!!\,(2([\frac{l}{2}]-l-2)\cdots(-l)\,(2k-1+l)!!}{(2k)!\,(2k+2[\frac{l}{2}]-l)\cdots(l-3)\cdots2\,(2k-l)} \\ a_{2k+1}=\dfrac{\prod_{i=1}^{k}(2i-1-l)\prod_{i=1}^{k}(2i+l)}{(2k+1)!}a_1 \end{cases}$$

$$\tilde{H}(x)=\sum_{k=0}^{\infty}a_{2k}x^{2k}+\sum_{k=0}^{\infty}a_{2k+1}x^{2k+1}$$

$$=a_0y_0(x)+a_1y_1(x)$$

l 为偶 $\Rightarrow$ $y_0(x)$ 为多项式 取 $a_1=0$

l 为奇 $\Rightarrow$ $y_1(x)$ 为多项式 取 $a_0=0$

l 非整数 $\Rightarrow$ $y_0(x)$, $y_1(x)$ 为发散无穷级数。

发散

对 y_0. $\frac{a_{2k+2}}{a_{2k}}>\frac{w_{k+1}}{w_k}$

令 $w_k:=\frac{1}{k\ln k}$

$\sum_{k=0}^{\infty}w_k>\int_2^{\infty}\frac{1}{x\ln x}dx$ 发散.

正则奇点

$$p(x)=\sum_{k=-1}^{\infty}p_k(x-x_0)^k$$

$$q(x)=\sum_{k=-2}^{\infty}q_k(x-x_0)^k$$

$\Rightarrow x=x_0$ 为 $y''(x)+p(x)y'(x)+q(x)y(x)=0$ 的正则奇点

作洛朗级数展开.

（且非本性奇点）

仅为一阶 or 二阶极点.

设 $y(x)=\sum_{k=0}^{\infty}a_k(x-x_0)^{k+s}$

$y'(x)=\sum_{k=0}^{\infty}a_k(k+s)(x-x_0)^{k+s-1}$

$y''(x)=\sum_{k=0}^{\infty}a_k(k+s)(k+s-1)(x-x_0)^{k+s-2}$

$p(x)=\sum_{k=0}^{\infty}p_k(x-x_0)^{k-1}$

$q(x)=\sum_{k=0}^{\infty}q_k(x-x_0)^{k-2}$

$$x^2y''+x^2p(x)y'+x^2q(x)y=0$$

$\Downarrow$

最低幂次 $(x-x_0)^{s-2}\to(x-x_0)^s$

$$a_0s(s-1)(x-x_0)^s+a_0sp_0(x-x_0)^s+a_0q_0(x-x_0)^s=0$$

$a_0\neq0$

$$\Rightarrow\quad s^2+s(p_0-1)+q_0=0$$

$$s=\frac{1-p_0\pm\sqrt{(p_0-1)^2-4q_0}}{2}$$

$$s_1-s_2=\sqrt{(p_0-1)^2-4q_0}$$

$$s_1+s_2=1-p_0$$

① 若 s_1-s_2 非整数

$y_1(x)=\sum_{k=0}^{\infty}a_k(x-x_0)^{k+s_1}$

$y_2(x)=\sum_{k=0}^{\infty}b_k(x-x_0)^{k+s_2}$

② 若 s_1-s_2 为整数

y_1,y_2 则可能非独立.

$\Rightarrow$ 独立解

$$y_2=Ay_1\ln(x-x_0)+\sum_{k=0}^{\infty}b_k(x-x_0)^{k+s_2}$$

若 $A=0$，则 y_1,y_2 和① 等价

若拓展至非正则奇点

$\Downarrow$

则不一定能找到第二个解（?）

Legendre 方程

$$r^2R''(r)+2rR'(r)-l(l+1)R(r)=0$$

$\Downarrow$

$$R''+\frac{2}{r}R'-\frac{l(l+1)}{r^2}R=0$$ ，$x=0$ 为正则奇点

$p_0=2\quad q_0=-l(l+1)$

$$\begin{cases}s_1=l\\ s_2=-(l+1)\end{cases}\Rightarrow\begin{cases}y_1(r)=\sum_{k=0}^{\infty}a_kr^{k+l}\\ y_2(r)=\sum_{k=0}^{\infty}b_kr^{k-l-1}\end{cases}$$

$$\begin{cases}r^2y_1''+2ry_1'-l(l+1)y_1=0\\ r^2y_2''+2ry_2'-l(l+1)y_2=0\end{cases}$$

$$a_k(k+l)(k+l-1)+2(k+l)a_k-l(l+1)a_k=0$$

$$a_k\left[k^2+(2l+1)k\right]=0\Rightarrow a_k=0$$

$$b_k(k-l-1)(k-l-2)+2(k-l-1)b_k-l(l+1)b_k=0$$

$$b_k\left[k^2-(2l+1)k\right]=0\Rightarrow b_k=0$$

$\Downarrow$

$$R(r)=a_0r^l+b_0\frac{1}{r^{l+1}}$$

λ为整数 $\Longleftrightarrow$ 包含z轴.

$$\rho^2 R''(\rho) + \rho R'(\rho) + (\mu\rho^2 - m^2) R(\rho) = 0$$

令 $x = \sqrt{\mu}\rho$

$$\Rightarrow x^2 R''(x) + x R'(x) + (x^2 - m^2) R(x) = 0$$

$$R''(x) + \frac{1}{x} R'(x) + \left(1 - \frac{m^2}{x^2}\right) R(x) = 0$$

$$p_{-1} = 1 \quad \begin{cases} q_{-2} = -m^2 \\ q_0 = 1 \end{cases} \Rightarrow \begin{cases} s_1 = \dfrac{1 - p_{-1} + \sqrt{(p_{-1} - 1)^2 - 4q_{-2}}}{2} = m \\ s_2 = -m \end{cases}$$

1) m 为非整数

$$\begin{cases} y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+m} \\ y_2(x) = \sum_{k=0}^{\infty} b_k x^{k-m} \end{cases}$$

$$x^2 y_1'' + x y_1' + (x^2 - m^2) y_1 = 0$$

$$\begin{cases} (m+1)(m) a_1 + (m+1) a_1 - m^2 a_1 = 0, & (1+2m) a_1 = 0 \xRightarrow{\text{if } (2m+1) \neq 0} a_1 = a_3 = \cdots = a_{2k+1} = 0 \\ (m+2)(m+1) a_2 + (m+2) a_2 + a_0 - m^2 a_2 = 0, & a_2 = \dfrac{-a_0}{4(m+1)} \end{cases}$$

$$(m+k)(m+k-1) a_k + (m+k) a_k + a_{k-2} - m^2 a_k = 0, \quad a_k = \frac{-a_{k-2}}{k(2m+k)}$$

$$a_{2k} = \frac{-a_{2k-2}}{2k(2m+2k)}, \qquad a_4 = \frac{-a_2}{4(2m+4)} = \frac{(-1)^2 a_0}{4 \times 2 (2m+4)(2m+2)}$$

$$a_6 = \frac{(-1)^3 a_0}{6 \times 4 \times 2 (2m+6)(2m+4)(2m+2)}$$

$$\Rightarrow a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (m+k) \cdots (m+1)} = \frac{(-1)^k \Gamma(m+1)}{2^{2k} k! \Gamma(m+k+1)} \cdot a_0$$

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^m \Gamma(m+1)}{2^{2k+m} k! \Gamma(m+k+1)} a_0 x^{2k+m}$$

$$= \tilde{a}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k+m}$$

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k+m}$$

m阶 Bessel 函数

$$y_2(x) = \tilde{b}_0 J_{-m}(x) = \tilde{b}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(-m+k+1)} \left(\frac{x}{2}\right)^{2k-m}$$

$$R(x) = C_1 J_m(x) + C_2 J_{-m}(x)$$

$$R(\rho) = C_1 J_m(\sqrt{\mu}\rho) + C_2 J_{-m}(\sqrt{\mu}\rho)$$

(x_n 为 $J_m(x)$ 零点)

$$R(\rho_0) = 0 \Rightarrow J_m(\sqrt{\mu}\rho_0) = 0, \quad \mu_n = \left(\frac{x_n}{\rho_0}\right)^2 \Rightarrow J_m(\sqrt{\mu}\rho) = J_m\left(x_n \frac{\rho}{\rho_0}\right)$$

$\lim_{\rho \to 0} R(\rho)$ 有限 $\Rightarrow C_2 = 0$

2) m 为整数.

$$J_{-m}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-m+k+1)} \left(\frac{x}{2}\right)^{2k-m}$$

$$J_{-1}(x) = \underbrace{\frac{1}{\Gamma(0)}\left(\frac{x}{2}\right)^{-1}}_{\to 0} + \frac{(-1)^1}{1!\,\Gamma(1)}\left(\frac{x}{2}\right)^1 + \frac{(-1)^2}{2!\,\Gamma(2)}\left(\frac{x}{2}\right)^3$$

$$J_1(x) = \frac{1}{\Gamma(2)}\left(\frac{x}{2}\right)^1 + \frac{(-1)^1}{2!\,\Gamma(3)}\left(\frac{x}{2}\right)^3$$

线性相关

$\Rightarrow$ $y_1 = J_m(x)$

$$y_2 = A J_m(x)\ln x + \sum_{k=0}^{\infty} b_k x^{k-m}$$

$$y_2' = A J_m'(x)\ln x + A\frac{1}{x}J_m(x) + \sum_{k=0}^{\infty}(k-m) b_k x^{k-m-1}$$

$$y_2'' = A J_m''(x)\ln x + A J_m'(x)\left(\frac{2}{x}\right) + A\left(-\frac{1}{x^2}\right)J_m(x) + \sum_{k=0}^{\infty}(k-m)(k-m-1) b_k x^{k-m-2}$$

代入 $x^2 y_2'' + x y_2' + (x^2-m^2) y_2 = 0$

$$x^2 y_2'' = A x^2 J_m''(x)\ln x + 2A x J_m'(x) - A J_m(x) + \sum_{k=0}^{\infty}(k-m)(k-m-1) b_k x^{k-m}$$

$$x y_2' = A x J_m'(x)\ln x + A J_m(x) + \sum_{k=0}^{\infty}(k-m) b_k x^{k-m}$$

$$(x^2-m^2) y_2 = A(x^2-m^2) J_m(x)\ln x - \sum_{k=0}^{\infty} m^2 b_k x^{k-m} + \sum_{k=0}^{\infty} b_k x^{k-m+2}$$

$\Rightarrow$ 对 x^{-m}: $\quad 0 \cdot b_0 = 0$. b_0 任意

对 x^{-m+1}: $\quad (1-2m) b_1 = 0$, $b_1 = 0$

对 x^{m}: $\quad A \frac{m}{m! \cdot 2^m} + 0 \cdot b_{2m} + b_{2m-2} = 0$

$$A = -(m-1)! \cdot 2^m b_{k-2}$$

$\Rightarrow$ Def $y_2(x)$ as $N_m(x)$

$$N_m(x) \xrightarrow{x \to 0} \text{发散}$$

$$\begin{cases} y_1'' + p y_1' + q y_1 = 0 & ① \\ y_2'' + p y_2' + q y_2 = 0 & ② \end{cases}$$

$$① \times y_2 - ② \times y_1 = y_2(x) y_1''(x) - y_1(x) y_2''(x) + p(x)\left[y_1'(x) y_2(x) - y_2'(x) y_1(x)\right]$$

Def $\Delta(x) = y_1'(x) y_2(x) - y_2'(x) y_1(x)$

$$\frac{d\Delta(x)}{dx} = y_1''(x) y_2(x) - y_2''(x) y_1(x)$$

$$\Rightarrow \quad \frac{d\Delta(x)}{dx} + p(x)\Delta(x) = 0$$

$$\frac{d\Delta(x)}{\Delta(x)} = -p(x)\,dx$$

$$\ln \Delta(x) = -\int p(x)\,dx$$

$$\Delta(x) = \Delta_0 e^{-\int p(x)dx} \quad *$$

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_2' y_1 - y_1' y_2}{y_1^2} = -\frac{\Delta(x)}{y_1^2}$$

$$\frac{y_2}{y_1} = -\int \frac{\Delta(x)}{y_1^2}dx = -\int \frac{\Delta_0 e^{-\int p(x)dx}}{y_1^2}dx$$

$$y_2 = -y_1 \int \frac{\Delta_0 e^{-\int p(x)dx}}{y_1^2}dx$$

$$p(x) = p_{-1}(x-x_0)^{-1} + p_0 (x-x_0)^0 + p_1(x-x_0) + \cdots$$

$$\int p(x)\,dx = p_{-1}\ln(x-x_0) + T_1$$

$$s_1 - s_2 = h$$

$$s_1 + s_2 = 1 - p_{-1}$$

$$y_2 = -y_1 \int \frac{\Delta_0 e^{-\int p(x)dx}}{y_1^2}dx = -y_1 \int \frac{\Delta_0 e^{-T_1}(x-x_0)^{-p_{-1}}}{y_1^2}dx = -y_1 \int \frac{T_2 (x-x_0)^{-p_{-1}}}{y_1^2}dx = -y_1 \int (x-x_0)^{-p_{-1}-2s_1} T_3(x)\,dx$$

$$= -y_1 \int (x-x_0)^{-h-1} T_3(x)\,dx$$

if h 为整数

$$LHS = A y_1(x)\ln(x-x_0) + (x-x_0)^{s_1}(x-x_0)^{-h} T_4(x)$$

$$= A y_1(x)\ln(x-x_0) + \sum_{k=0}^{\infty} b_k (x-x_0)^{k+s_2}$$

非正则奇点

$$y''(x)+p(x)y'(x)+q(x)y(x)=0$$

$$y(x)=\sum_{k=0}^{\infty}a_k(x-x_0)^{k+s}$$

$$\underbrace{(x-x_0)^{s-2}+(x-x_0)^{s-1}(x-x_0)^{p_{min}}+(x-x_0)^{s}(x-x_0)^{q_{min}}}_{\text{可能无解}}$$

故用正常点展开来求解.

§9.4 Sturm-Liouville 本征值问题.

$$\frac{d}{dx}\left[k(x)\frac{dy}{dx}\right]-q(x)y(x)+\lambda\rho(x)y(x)=0$$

$$k(x)y''(x)+k'(x)y'(x)-q(x)y(x)+\lambda\rho(x)y(x)=0$$

$$\begin{cases}k(x)=e^{\int a(x)dx}\\ q(x)=-b(x)e^{\int a(x)dx}\\ \rho(x)=c(x)e^{\int a(x)dx}\end{cases}$$

加上第一类/第二类/第三类齐次边界条件

(包括自然条件)

可以构成 S-L 本征值问题.

$$y\big|_{x=a}=0\ /\ y'\big|_{x=b}=0\ /\ y+Hy'\big|_{x=L}=0$$

ep ① $a=0,\ b=l,\ k(x)=\text{const},\ q(x)=0,\ \rho(x)=\text{const}$

$$y''(x)+\lambda y(x)=0$$

$$\begin{cases}y|_{x=a}=0\\ y|_{x=b}=0\end{cases}\Rightarrow y=\sin\frac{n\pi}{l}x$$

$$\begin{cases}y|_{x=a}=0\\ y'|_{x=b}=0\end{cases}\Rightarrow y=\sin\frac{(n+\frac{1}{2})\pi}{l}x$$

③ $a=-1,\ b=+1,\ k(x)=1-x^2,\ q(x)=0,\ \rho(x)=1$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\lambda y=0 \qquad \begin{cases} y_1(x) \\ y_{-1}(x) \end{cases} \text{有限}$$

legendre 方程

if $q(x)=\frac{m^2}{1-x^2}$

$$(1-x^2)y''-2xy'+\left[\lambda-\frac{m^2}{1-x^2}\right]y(x)=0$$

缔合 legendre 方程.

性质:

① 若 $k(x), k'(x), q(x)$ 连续. or 在 $x=a, x=b$ 处有一阶极点，则存在无穷多本征值 λ_n

$(\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_n\le\cdots)$

对应无穷多个本征函数.

② 所有本征值 $\lambda_n\ge0$

③ 对不同本征值 λ_n, λ_m 对应的本征函数 $y_n\ y_m$ 在 (a,b) 上带权重 $\rho(x)$ 正交.

$$\int_a^b y_n(x)y_m(x)\rho(x)\,dx=0 \quad , \lambda_n\ne\lambda_m$$

$$\begin{cases} \frac{d}{dx}\left[k(x)\frac{dy_n}{dx}\right]-q(x)y_n(x)+\lambda_n\rho(x)y_n=0 & ① \\ \frac{d}{dx}\left[k(x)\frac{dy_m}{dx}\right]-q(x)y_m(x)+\lambda_m\rho(x)y_m=0 & ② \end{cases}$$

$①\times y_m(x)-②\times y_n(x)$ 再积分

$$\underbrace{\int_a^b y_m\frac{d}{dx}\left[k(x)\frac{dy_n}{dx}\right]-y_n\frac{d}{dx}\left[k(x)\frac{dy_m}{dx}\right]}_{=0}+\underbrace{(\lambda_n-\lambda_m)}_{\ne}\underbrace{\int_a^b y_n(x)y_m(x)\rho(x)\,dx}_{\Downarrow}=0$$

即 $\int_a^b\frac{d}{dx}\left[k(y_my_n'-y_m'y_n)\right]$ 由齐次边界条件 为0

④ 完备性

$$f(x)=\sum_{n=1}^{\infty}f_n y_n(x) \qquad f_n=\frac{\int_a^b f(x)y_n(x)\rho(x)dx}{\int_a^b y_n^2(x)\rho(x)dx}$$

复数形式

正交性： $\int_a^b y_n(x) y_m^*(x) \rho(x) dx$

$(N_n)^2 = \int_a^b y_n(x) y_n^*(x) \rho(x) dx$

模方

$f_n = \frac{1}{N_n^2} \int_a^b y_n(x) y_n^*(x) \rho(x) dx$

$\Phi'' + \lambda \Phi = 0$

$\Phi_m = e^{im\varphi}$

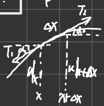
$\int_0^{2\pi} \Phi_m(\varphi) \Phi_n^*(\varphi) d\varphi = \int_0^{2\pi} e^{im\varphi} e^{-in\varphi} d\varphi = 2\pi \delta_{m,n}$

$f(\varphi) = \sum_{-\infty}^{\infty} f_n e^{in\varphi}$, $f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) e^{-in\varphi} d\varphi$

一、波动方程.

弦振动:

1) 无受迫.

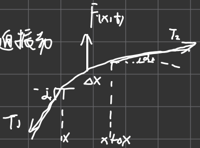


$T_2\sin\alpha_2 - T_1\sin\alpha_1 = \rho\Delta x\, u_{tt}$

$= T_2\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - T_1\frac{\partial u}{\partial x}\Big|_x$

$\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = 0$

2) 受迫振动



$F(x,t)\Delta x + T_2\sin\alpha_2 - T_1\sin\alpha_1 = \rho\Delta x\, u_{tt}$

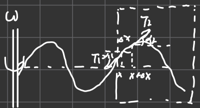
$\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho}F(x,t) \equiv f(x,t)$

3) 阻尼振动.

$f(x,t) = \frac{F(x,t)}{\rho} = -\frac{1}{\rho}R\frac{\partial u}{\partial t} \overset{def}{\equiv} -2b\frac{\partial u}{\partial t}$ 阻尼系数

$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial u}{\partial t} = 0$

纵向具有加速度



$T_1\cos\alpha_1 - T_2\cos\alpha_2 = \rho\Delta x\,\omega^2 x$

$\overset{\alpha\to 0}{\Rightarrow} T|_x - T|_{x+\Delta x} = \rho\Delta x\,\omega^2 x$

$\frac{\partial T}{\partial x} = -\rho\omega^2 x$

$T = -\int_l^x \rho\omega^2 x\,dx = \frac{1}{2}\rho\omega^2 l^2 - \frac{1}{2}\rho\omega^2 x^2$

$a^2 = \frac{T}{\rho} = \frac{1}{2}\omega^2(l^2 - x^2)$

$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\omega^2(l^2-x^2)\frac{\partial^2 u}{\partial x^2} = 0$

初始条件

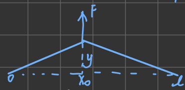
$u_{tt} - a^2 u_{xx} = 0$

$\begin{cases} u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$

→ 初始弦位移

$T\frac{y}{x} + \frac{y}{l-x_0} = F$

$y = \frac{F}{T}\frac{x_0(l-x_0)}{l}$

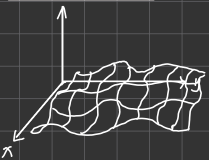


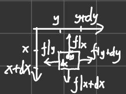
$u = \begin{cases} \frac{F}{T}\frac{l-x_0}{l}x & 0\le x< x_0 \\ \frac{F}{T}\cdot\frac{x_0}{l}(l-x) & x_0<x<l \end{cases}$

→ 初始冲量.

$u_t = \begin{cases} 0 & |x-x_0|>\delta \\ \frac{1}{\rho}I\delta(x-x_0) & |x-x_0|<\delta \end{cases}$

平面振动





$(F|_{x+dx} - F|_x)dy \approx 0$

$(F|_{y+dy} - F|_y)dx \approx 0$

$F|_{x+dx}dy\cdot u_x|_{x+dx} - F|_x dy\, u_x|_x + F|_{y+dy}dx\, u_y|_{y+dy} - F|_y dx\, u_y|_y = \rho\,dx\,dy\,u_{tt}$

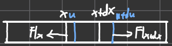
$F_x u_{xx} + F_y u_{yy} = \rho u_{tt}$

$F_x = F_y$

$\Rightarrow u_{tt} - a^2\Delta u = 0,\quad a^2 = \frac{F}{\rho} \equiv f$

杆纵振动

1) 无受迫



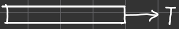
$F|_{x+dx} - F|_x = YS\,u_x|_{x+dx} - YS\,u_x|_x = \rho S\,dx\,u_{tt}$

$u_{tt} - a^2 u_{xx} = 0$

$a^2 = \frac{Y}{\rho}$

(2) 受迫振动

端受迫



$u_{tt} - a^2 u_{xx} = 0$

考虑边界条件.

$F|_{l-dx} \leftarrow \square \rightarrow T$

$T = YS\, u_x|_{x=l}$

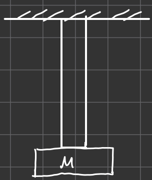
$u_x|_{x=l} = \frac{T}{YS}$

$$\begin{cases} u|_{x=0} = 0 \\ u_x|_{x=l} = \frac{T}{YS} \end{cases}$$

均匀受迫

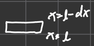
$u_{tt} - a^2 u_{xx} = f(x,t)$

非典型边界条件



$u_{tt} - a^2 u_{xx} = 0$

$$\begin{cases} u|_{t=0} = 0 \\ u_t|_{t=0} = 0 \\ u|_{x=0} = 0 \\ \left(u_{tt} - g + \frac{YS}{M} u_x\right)\Big|_{x=l} = 0 \end{cases}$$



$Mg - F|_{x=l} = M u_{tt}|_{x=l}$

$F|_{x=l} - F|_{x=l-dx} = YS\left(u_x|_{x=l} - u_x|_{x=l-dx}\right) = \rho S dx\, u_{tt}$

$dx \to 0, \quad F|_{x=l} = M(g - u_{tt})|_{x=l} = YS\, u_x|_{x=l}$

$\left(u_{tt} - g + \frac{YS}{M} u_x\right)\Big|_{x=l} = 0$

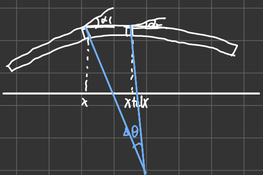
挠度 $\frac{d^2\omega}{dx^2} = \frac{M}{YI} = u_{xx}$, $M = YI u_{xx}$

简化 $M|_x - M|_{x+dx} - T dx = 0$, $T = -\frac{\partial M}{\partial x} = -YI u_{xxx}$.

$T|_x - T|_{x+dx} = \rho dx\, u_{tt}$

$YI u_{xxxx} = \rho u_{tt}$, $\quad u_{tt} - \frac{YI}{\rho} u_{xxxx} = 0$

杆横振动:



$tg\alpha \approx \alpha = u_x \quad (\alpha \to 0)$

$|\Delta\theta| = |\alpha_2 - \alpha_1| = \frac{\sqrt{(\Delta x)^2 + (\Delta u)^2}}{R} = \frac{[1+(u_x)^2]^{1/2}\Delta x}{R}$

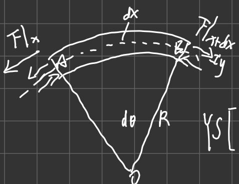
曲率半径

$\Rightarrow \frac{d\theta}{dx} = [1+(u_x)^2]^{1/2}/R = \frac{d\alpha}{dx}$

$tg\alpha = u_x$

$\frac{\partial tg\alpha}{\partial x} = u_{xx} = \frac{\partial}{\partial x}\left(\frac{\sin\alpha}{\cos\alpha}\right) = (1+tg^2\alpha)\frac{\partial\alpha}{\partial x} = [1+(u_x)^2] \cdot \frac{[1+(u_x)^2]^{1/2}}{R}$

$R = \frac{[1+(u_x)^2]^{3/2}}{u_{xx}}$, $\quad \frac{d\theta}{dx} = \frac{d\alpha}{dx} = \frac{u_{xx}}{1+(u_x)^2}$



$YS\left[\frac{(R+y)d\theta - dx}{dx}\right] = YS\left[\frac{(R+y)\frac{dx}{R} - dx}{dx}\right] = YS\cdot\frac{y}{R} = F$

$S = b\,dy$

$F = \frac{Yb}{R} y\,dy$

F对A点矩: $\quad F_y = \frac{Yb}{R} y^2 dy$.

总力矩 $M = \frac{Y}{R}\int b y^2 dy = \frac{Y}{R} I_c = \frac{YI_c u_{xx}}{[1+(u_x)^2]^{3/2}} \xlongequal{u_x^2 \to 0} YI_c u_{xx}$

力矩平衡

$M|_{x+dx} - M|_x + F\Delta x = 0$

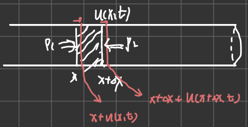
$F = -\frac{\partial}{\partial x} M = -YI_c u_{xxx}$

$F|_x - F|_{x+dx} = \rho\Delta x\, u_{tt}$

$YI_c u_{xxxx} = \rho u_{tt}$

$u_{tt} - \frac{YI_c}{\rho} u_{xxxx} = 0$

声音的传播.



主体运动 → ρ改变 → P改变

压强 $p = f(\rho)$ $p_0 = f(\rho_0)$

$$\begin{cases} p = f(\rho) = p_0 + \Delta p \\ p_0 = f(\rho_0) \end{cases}$$

$$\underbrace{f(\rho_0+\Delta\rho)}_{p} = \underbrace{f(\rho_0)}_{p_0} + \underbrace{f'(\rho_0)\Delta\rho}_{\Delta p}$$

$1\ \text{bar} = 10^5\ N/m$

$L = 20\log_{10}\left(\frac{\Delta p}{p_{ref}}\right)$

$P_{ref} = 2\times 10^{-10}\ \text{bar}$

$$P_1 S - P_2 S = \rho_0 \cdot S \cdot \Delta x \cdot \frac{\partial^2}{\partial t^2} u(x,t)$$

$$\Delta P_1 - \Delta P_2 = \rho_0 \Delta x\, u_{tt}(x,t)$$

$$f'(\rho_0)\Delta\rho\Big|_{x=x} - f'(\rho_0)\Delta\rho\Big|_{x=x+\Delta x} = \rho_0 \Delta x\, u_{tt}(x,t) \quad ①$$

$$\rho_0 \Delta x S = \rho(\Delta x + u(x+\Delta x,t) - u(x,t))S$$

$$\Rightarrow \quad \rho_0 = \rho + \rho\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} = \rho + \rho\frac{\partial u}{\partial x}(x,t)$$

$$\Rightarrow \quad \Delta\rho\Big|_{x=x} = -\rho\frac{\partial u}{\partial x}(x,t) = -\rho_0\frac{\partial u}{\partial x}(x,t)$$

$$\Delta\rho\Big|_{x=x+\Delta x} = -\rho\frac{\partial u}{\partial x}(x+\Delta x,t) = -\rho_0\frac{\partial u}{\partial x}(x+\Delta x,t)$$

$$① \Rightarrow f'(\rho_0)\rho_0\left[\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right] = \rho_0 \Delta x\, u_{tt}(x,t)$$

$$\Rightarrow \quad f'(\rho_0)\frac{\frac{\partial u}{\partial x}\big|_{x+\Delta x} - \frac{\partial u}{\partial x}\big|_{x}}{\Delta x} = u_{tt}(x,t)$$

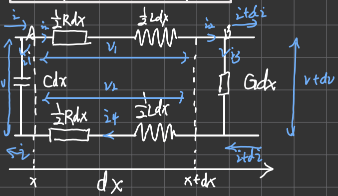
$$\frac{\partial^2 u}{\partial t^2} - f'(\rho_0)\cdot\frac{\partial^2 u}{\partial x^2} = 0$$

$$f'(\rho_0) = \frac{\partial p}{\partial \rho}\Big|_{\rho=\rho_0} = K = C_s^2$$

$$\Downarrow$$

$$\frac{\partial^2 u}{\partial t^2} - K\frac{\partial^2 u}{\partial x^2} = 0$$

电输运 or 电报方程



① at A: $i = i_1 + i_2$

at B: $i_2 = i_3 + i + di = i_1 + i_3 + di + i_2$

$i_1 + i_3 + di = 0$

$i_1 = \frac{dQ}{dt} = Cdx\frac{\partial v}{\partial t}$, $i_3 = Gdx(v+dv) = Gdx \cdot v$

$di = -Cdx\frac{\partial v}{\partial t} - Gdx\, v$, $i_2 = i - Cdx\frac{\partial v}{\partial t}$

$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} - Gv$ □

② From A to A (CW)

$v_1 + v + dv + v_2 - v = 0$

$v_1 + v_2 + dv = 0$

$v_1 = i_2\frac{1}{2}Rdx + \frac{1}{2}Ldx\frac{\partial i_2}{\partial t}$, $v_2 = i_4\frac{1}{2}Rdx + \frac{1}{2}Ldx\frac{di_4}{dt}$

$= \frac{i}{2}Rdx + \frac{1}{2}Ldx\frac{\partial i}{\partial t}$ $= \frac{i}{2}Rdx + \frac{1}{2}Ldx\frac{\partial i}{\partial t}$

$iRdx + Ldx\frac{\partial i}{\partial t} + dv = 0$ $\frac{\partial v}{\partial x} = -iR - L\frac{\partial i}{\partial t}$ △

$$\begin{cases} \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} - Gv & □ \\ \frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} - Ri & △ \end{cases}$$

$$\begin{cases}\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} - Gv & ① \\ \frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} - Ri & ②\end{cases}$$

$$\begin{cases}\frac{\partial^2 i}{\partial x^2} = -C\frac{\partial^2 v}{\partial t\partial x} - G\frac{\partial v}{\partial x} \\ \frac{\partial^2 v}{\partial x\partial t} = -L\frac{\partial^2 i}{\partial t^2} - R\frac{\partial i}{\partial t}\end{cases} \Rightarrow \frac{\partial^2 i}{\partial x^2} = CL\frac{\partial^2 i}{\partial t^2} + CR\frac{\partial i}{\partial t} - G\frac{\partial v}{\partial x} = CL\frac{\partial^2 i}{\partial t^2} + (CR+GL)\frac{\partial i}{\partial t} + GRi$$

$$\begin{cases}\frac{\partial^2 v}{\partial x^2} = -L\frac{\partial^2 i}{\partial t\partial x} - R\frac{\partial i}{\partial x} \\ \frac{\partial^2 i}{\partial x\partial t} = -C\frac{\partial^2 v}{\partial t^2} - G\frac{\partial v}{\partial t}\end{cases} \Rightarrow \frac{\partial^2 v}{\partial x^2} = CL\frac{\partial^2 v}{\partial t^2} + GL\frac{\partial v}{\partial t} - R\frac{\partial i}{\partial x} = CL\frac{\partial^2 v}{\partial t^2} + (GL+RC)\frac{\partial v}{\partial t} + GRv$$

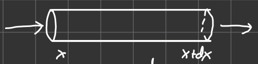
$$\Downarrow$$

$$u_{tt} - a^2 u_{xx} + 2b u_t + cu = 0$$

忽略 G、R $\Rightarrow$ $u_{tt} - a^2 u_{xx} = 0 \quad (u = i \text{ or } v) \quad a = \frac{1}{\sqrt{CL}}$

二、输运方程.

$\boxed{\text{热传导方程.}}$



有热源: $u_t - a^2 u_{xx} = f(x,t)$

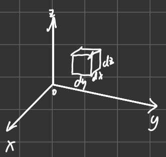
$Q_{in} = -kS\Delta t \frac{\partial u}{\partial x}\Big|_x$ | $Q_{out} = -kS\Delta t \frac{\partial u}{\partial x}\Big|_{x+dx}$

$$Q_{in} - Q_{out} = c\rho S\, dx\, \Delta u$$

$$k\frac{\partial^2 u}{\partial x^2} = c\rho\frac{\partial u}{\partial t} \Rightarrow u_t - \frac{k}{c\rho}u_{xx} = 0$$

$$u_t - a^2 u_{xx} = 0, \quad a^2 = \frac{k}{c\rho}$$

$\boxed{\text{中子裂变}}$



$n +$ 铀 $\rightarrow$ 碎块 $+ \beta n$

$$dN = \Gamma|_x\, dy\,dz\,dt + \Gamma|_y\, dx\,dz\,dt + \Gamma|_z\, dx\,dy\,dt - \left(\Gamma|_{x+dx}\, dy\,dz\,dt + \Gamma|_{y+dy}\, dx\,dz\,dt + \Gamma|_{z+dz}\, dx\,dy\,dt\right) + \beta u\, dx\,dy\,dz\,dt = \frac{\partial u}{\partial t}\, dx\,dy\,dz\,dt$$

$$\Rightarrow \frac{\partial u}{\partial t} = -\frac{\partial \Gamma}{\partial x} - \frac{\partial \Gamma}{\partial y} - \frac{\partial \Gamma}{\partial z} + \beta u = D\frac{\partial^2 u}{\partial x^2} + D\frac{\partial^2 u}{\partial y^2} + D\frac{\partial^2 u}{\partial z^2} + \beta u = \beta u + D\Delta u$$

$$\boxed{\frac{\partial u}{\partial t} - D\Delta u = \beta u}$$

- Fission $\beta > 0$
- uniform $\beta = 0$ No origin/sink
- attenuation/decrease/衰减. $\beta < 0$

or

$$\boxed{\begin{gathered}\frac{\partial u}{\partial t} - \nabla(D\nabla u) = \beta u \\ \Rightarrow \frac{\partial u}{\partial t} - (\nabla D)(\nabla u) - D\Delta u = \beta u\end{gathered}}$$

三、稳定场分布.

Maxwell Equation
$$\begin{cases}\nabla\cdot\vec{D}=\rho\\ \nabla\cdot\vec{B}=0\\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}\end{cases}$$

Material Equation
$$\begin{cases}\vec{D}=\varepsilon_0\vec{E}\\ \vec{B}=\mu_0\vec{H}\\ \vec{J}=\sigma\vec{E}\end{cases}$$

① $\nabla\times(\nabla\times E)=\nabla(\nabla\cdot E)-\nabla^2 E$

I. $\nabla\times(\nabla\times E)=\nabla\times(-\frac{\partial}{\partial t}B)=-\frac{\partial}{\partial t}(\nabla\times B)=-\mu_0\frac{\partial}{\partial t}(\nabla\times H)=-\mu_0\frac{\partial}{\partial t}[J+\frac{\partial D}{\partial t}]$

$\mu_0\sigma\frac{\partial E}{\partial t}\quad\mu_0\varepsilon_0\frac{\partial E}{\partial t^2}$

II. $\nabla(\nabla\cdot E)=\frac{1}{\varepsilon_0}\nabla(\nabla\cdot D)=\frac{1}{\varepsilon_0}\nabla\rho\ \underset{\rho=const}{\overset{assume}{=}}\ 0$

III. $\nabla^2 E=\Delta E$

$$\Rightarrow\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}+\mu_0\sigma\frac{\partial E}{\partial t}=\Delta E$$

② $\nabla\times(\nabla\times H)=\nabla(\nabla\cdot H)-\nabla^2 H$

I. $\nabla\times(\nabla\times H)=\nabla\times(J+\frac{\partial D}{\partial t})=\nabla\times J+\varepsilon_0\frac{\partial}{\partial t}\nabla\times E=(\sigma+\varepsilon_0\frac{\partial}{\partial t})(\nabla\times E)$

$=-\mu_0\sigma\frac{\partial H}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2 H}{\partial t^2}$

II. $\nabla(\nabla\cdot H)=\frac{1}{\mu_0}\nabla(\nabla\cdot B)=0$

III. $\nabla^2 H=\Delta H$

$$\Rightarrow\mu_0\varepsilon_0\frac{\partial^2 H}{\partial t^2}+\mu_0\sigma\frac{\partial H}{\partial t}=\Delta H$$

when $\sigma=0$ (电磁波损耗为0)

then
$$\begin{cases}\frac{\partial^2 E}{\partial t^2}-c^2\nabla^2 E=0\\ \frac{\partial^2 H}{\partial t^2}-c^2\nabla^2 H=0\end{cases}$$