剪应力 shear stress $\tau = \mu \frac{du}{dy}$ $\quad \tau = \frac{F}{A}$

流线 stream line $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

流体受的力

$$\begin{cases} \text{压力 press } f_p = -\nabla p = -(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}) \\ dF_p = (-\vec{i}\frac{\partial p}{\partial x} - \vec{j}\frac{\partial p}{\partial y} - \vec{k}\frac{\partial p}{\partial z})\,dx\,dy\,dz \\ \text{黏性力 viscosity } f_{vs} = \mu\nabla^2 V = \mu(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}) \\ \text{重力 gravity } f_{grav} = \rho\vec{g} \end{cases}$$

流体力学中牛顿第二定律：$\nabla p = \rho(\vec{g} - \vec{a}) + \mu\nabla^2 V$

一、积分法

雷诺转换公式：$\frac{dB}{dt}\Big|_{sys} = \frac{\partial}{\partial t}\int_{cv}\beta\rho\,dV + \int_{A_e}\beta_e\rho_e V_e\,dA_e - \int_{A_i}\cancel{\beta_e\rho}\,\beta_i\rho_i V_i\,dA_i$

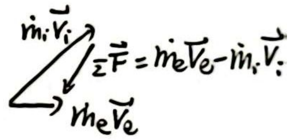
$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{cv}\beta\rho\,dV\right) + \int_{cs}\beta\rho(\vec{V_r}\cdot\vec{n})\,dA$$

① 质量守恒：$\frac{dm}{dt}\Big|_{sys} = \frac{\partial}{\partial t}\int_{cv}\rho\,dV + \int_{A_e}\rho_e\vec{V_e}\,dA_e - \int_{A_i}\rho_i\vec{V_i}\,dA_i = 0$

$$\frac{dm}{dt}\Big|_{cv} + \sum_{出}\dot{m}_e - \sum_{进}\dot{m}_i = 0.$$

稳态时 $\frac{dm}{dt}\Big|_{cv} = 0$ $\quad \sum\dot{m}_{in} = \sum\dot{m}_{out}$

② 动量定律：$\sum\vec{F} = \frac{\partial}{\partial t}\int_{cv}\vec{V}\rho\,dV + \int_{A_e}\vec{V}\,d\dot{m}_e - \int_{A_i}\vec{V}\cdot d\dot{m}_i$



$$= \frac{\partial}{\partial t}\int_{cv}\vec{V}\rho\,dV + \sum(\dot{m}_e V_e)_{out} - \sum(\dot{m}_i V_i)_{in}.$$

动量修正修数 $\beta = \frac{1}{A}\int\left(\frac{u}{V_{av}}\right)^2 dA = \begin{cases} \text{层流 laminar flow} \quad \frac{4}{3} \\ u = u_0(1-\frac{r^2}{R^2}) \\ \text{湍流 Turbulent flow} \quad \frac{(1+m)^2(2+m)^2}{2(1+2m)(2+2m)} \\ u \approx u_0(1-\frac{r}{R})^m \end{cases}$

③ 能量守恒. $\left(\frac{p}{\rho g} + \frac{\alpha}{2g}V^2 + z\right)_{in} = \left(\frac{p}{\rho g} + \frac{\alpha}{2g}V^2 + z\right)_{out} + h_{turbine} + h_{friction} - h_{pump}$

涡轮机 摩擦 泵

能量修正系数 $\alpha = \frac{1}{A}\int\left(\frac{u}{V_{av}}\right)^3 dA$

$$= \begin{cases} \text{层流} \quad 2 \\ \text{湍流} \quad \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)} \end{cases}$$

伯努利方程：$\frac{P_1}{\rho}+\frac{1}{2}V_1^2+gz_1=\frac{P_2}{\rho}+\frac{1}{2}V_2^2+gz_2=const$

└> 使用条件
① 流线上2点.
② 非黏性流体
③ 不可压缩
④. no shaft work 没有 turbine, pump.

二.微元法

$\vec{V}(\vec{r},t)=\vec{i}\,u(x,y,z,t)+\vec{j}\,v(x,y,z,t)+\vec{k}\,w(x,y,z,t)$.

$\vec{a}=\frac{\partial V}{\partial t}+(\vec{V}\cdot\nabla)V=\frac{\partial V}{\partial t}+(u\frac{\partial V}{\partial x}+v\frac{\partial V}{\partial y}+w\frac{\partial V}{\partial z})$.

①、质量守恒：连续性方程.

直角坐标：$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$

极坐标：$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)=0$.

稳态时$\frac{\partial \rho}{\partial t}=0$. 不可压缩时 $\rho=const$ $\frac{\partial \rho}{\partial t}\approx 0$.

方程可化简为
$$\begin{cases}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{V}=0\\ \frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta)+\frac{\partial}{\partial z}(V_z)=0.\end{cases}$$

②、动量定律.

不可压缩的牛顿流体 $\rho.\ \mu=const$. 有 Navier Stokes equations NS方程.

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\rho\frac{du}{dt}$$

$$\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=\rho\frac{dv}{dt}$$

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\rho\frac{dw}{dt}$$

① stream function 流函数

- 直角坐标中
  - 不可压缩流体：$u=\frac{\partial\psi}{\partial y}$ $v=-\frac{\partial\psi}{\partial x}$ $d\psi=$ 2条流线间体积流量
  - 可压缩流体：$\rho u=\frac{\partial\psi}{\partial y}$ $\rho v=-\frac{\partial\psi}{\partial x}$ $d\psi=\dot{m}_{1\to2}$
- 极坐标中（不可压缩的）
  - ① $V_z=0$. $V_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$ $V_\theta=-\frac{\partial\psi}{\partial r}$
  - ② $V_\theta=0$ $V_r=-\frac{1}{r}\frac{\partial\psi}{\partial z}$ $V_z=\frac{1}{r}\frac{\partial\psi}{\partial r}$

vorticity and irrotationality 有旋、无旋流动

$$\omega=\vec{i}\,\omega_x+\vec{j}\,\omega_y+\vec{k}\,\omega_z=\frac{1}{2}(\mathrm{curl}\,\vec{V})=\frac{1}{2}\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ u&v&w\end{vmatrix}=\cancel{\frac{1}{2}\begin{vmatrix}\vec{i}_r&\vec{i}_\theta&\vec{k}\\ \frac{1}{r}\frac{\partial}{\partial r}&\frac{1}{r}\frac{\partial}{\partial\theta}&\frac{\partial}{\partial z}\\ V_r&V_\theta&V_z\end{vmatrix}}$$

$$\begin{cases}2\omega_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\\ 2\omega_y=\cancel{\frac{\partial w}{\partial x}}\ \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\\ 2\omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\end{cases}\qquad\begin{cases}2\omega_r=\frac{1}{r}\frac{\partial V_z}{\partial\theta}-\frac{\partial V_\theta}{\partial z}\\ 2\omega_\theta=\frac{\partial V_r}{\partial z}-\frac{\partial V_z}{\partial r}\\ 2\omega_z=\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)-\frac{1}{r}\frac{\partial V_r}{\partial\theta}\end{cases}$$

$\omega=0$ 无旋 $\omega\neq0$ 有旋

黏性流体是有旋的，非黏性流体是无旋的，伯努利方程只能用于非黏性流体，且能用于无旋流体.

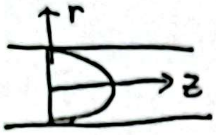
② velocity potential function 势函数.

如果是无旋流体，$\mathrm{curl}\,\vec{V}=0$. 可定义出一个新函数 ϕ $\phi\perp\psi$

- 直角坐标：$u=\frac{\partial\phi}{\partial x}$ $v=\frac{\partial\phi}{\partial y}$ $w=\frac{\partial\phi}{\partial z}$
- 极坐标：$V_r=\frac{\partial\phi}{\partial r}$ $V_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

圆管内

一、层流



① $V_z=(-\frac{dp}{dz})\frac{1}{4\mu}(R^2-r^2)$

$V_z=(-\frac{dp}{dz})\frac{1}{4\mu}(R^2-r^2)$

推导：列连续性方程化简得 $\frac{\partial V_z}{\partial z}=0$. $\Rightarrow V_z=V_z(r)$ only

列NS方程化简得 $\frac{\mu}{r}\frac{d}{dr}(r\frac{dV_z}{dr})=\frac{dp}{dz}=const$

$\Rightarrow V_z=\frac{dp}{dz}\frac{r^2}{4\mu}+c_1\ln r+c_2$

$r=R$ 时 $V_z=0$；$r=0$ 时 $V_z=$ finite $\Rightarrow V_z=(-\frac{dp}{dz})\frac{1}{4\mu}(R^2-r^2)$

$V_{max}=(-\frac{dp}{dz})\frac{R^2}{4\mu}$ $\qquad -\frac{dp}{dz}=\frac{\Delta p}{L}$

$V_{avg}=(-\frac{dp}{dz})\frac{R^2}{8\mu}$ $\qquad \tau_{wall}=\mu\left|\frac{\partial V_z}{\partial r}\right|_{r=R}=(-\frac{dp}{dz})\frac{R}{2}$

$Q=(-\frac{dp}{dz})\frac{\pi R^4}{8\mu}$

V_{max}、V_{avg}、Q、τ_{wall}、$\frac{\Delta p}{L}$ 可以相互推导.

如：$Q=\frac{\pi R^4}{8\mu}\frac{\Delta p}{L}\Rightarrow\frac{\Delta p}{L}=\frac{8Q\mu}{\pi R^4}$ $\qquad \tau_w=\frac{R}{2}\frac{\Delta p}{\Delta L}\Rightarrow\Delta p=\frac{2\tau_w\Delta L}{R}$

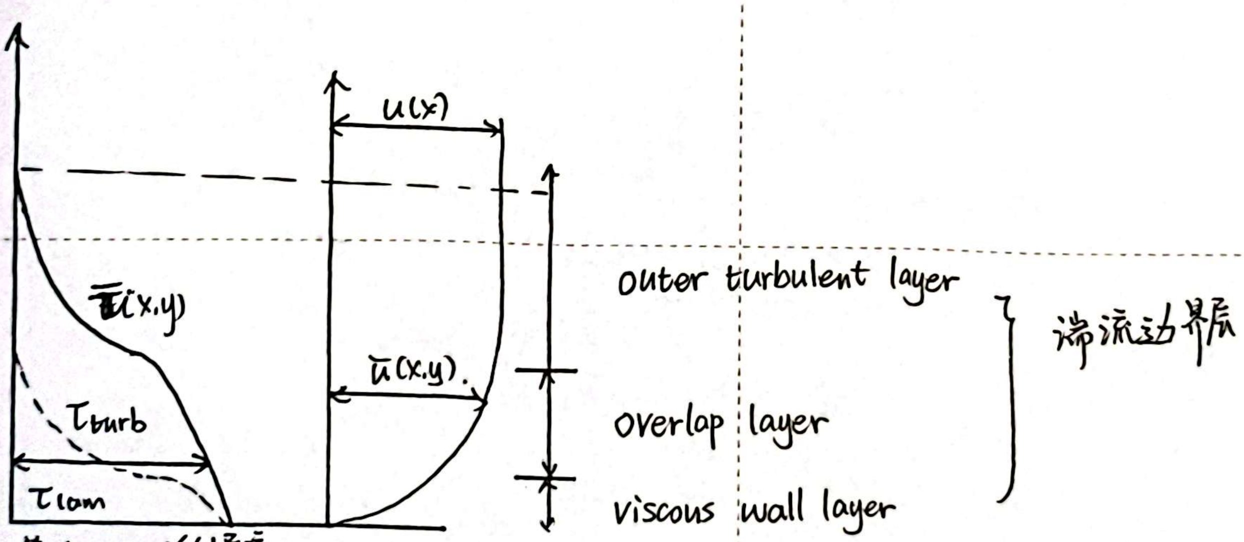
二、湍流

$\frac{u}{u^*}=\frac{1}{k}\ln\frac{yu^*}{\nu}+B=\frac{1}{0.41}\ln\frac{yu^*}{\nu}+5$.

在圆管内有：$\frac{V}{u^*}=2.44\ln\frac{Ru^*}{\nu}+1.34$.

friction velocity $u^*=(\frac{\tau_w}{\rho})^{\frac{1}{2}}$ (定义式). (称作摩擦速度，但并不是速度，只是量纲为LT^{-1}).

$\hookrightarrow \tau_w=\rho u^{*2}$

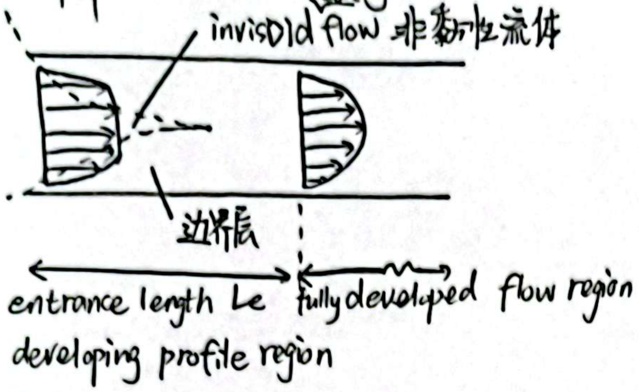


靠近地面 近似层流

湍流中 $\bar{u}=\frac{1}{T}\int_0^T u\,dt$

湍流中NS方程：$\rho\frac{d\bar{u}}{dt}=-\frac{\partial\bar{p}}{\partial x}+\rho g_x+\frac{\partial\tau}{\partial y}$ $\qquad \tau=\mu\frac{\partial\bar{u}}{\partial y}-\rho\overline{u'v'}=\tau_{lam}+\tau_{turb}$.

计算 entrance length



$$\frac{L_e}{d} = 0.06\, Re_d \quad \text{laminar} \quad Re<2300$$

$$\frac{L_e}{d} = 4.4\, Re_d^{\frac{1}{6}} \quad \text{turbulent} \quad Re>2300.$$

阻力系数 darcy friction factor

①定义式: $f = \dfrac{(\frac{D}{L})\Delta p_f}{\rho \frac{V^2}{2}}$ Δp_f 为摩擦引起的压降.

$$\Delta p_f = \rho g h_f = \frac{f \cdot \rho \frac{V^2}{2}}{(\frac{D}{L})} \Rightarrow h_f = f \frac{L}{D} \frac{V^2}{2g} \quad (\text{friction head loss})$$

② $f = \dfrac{8\tau_w}{\rho V^2}$ (对层流和湍流都适用). $\Rightarrow (\tau_w = \frac{f}{8}\rho V^2)$

推导: 质量守恒: $Q_1 = Q_2 \Rightarrow V_1 = V_2$

能量方程: $\frac{p_1}{\rho g} + \frac{\alpha V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\alpha V_2^2}{2g} + z_2 + h_f \Rightarrow h_f = \frac{\Delta p}{\rho g} + \Delta z$

动量方程: $\Sigma F = \Delta p(\pi R^2) + \rho g \pi R^2 \underbrace{L\sin\phi}_{=\Delta z} - \tau_w(2\pi R)L = m(V_2 - V_1) = 0$

$$\Rightarrow \frac{\Delta p}{\rho g} + \Delta z = \frac{2\tau_w L}{\rho g R} = \frac{4\tau_w}{\rho g}\frac{L}{d}$$

$$\Rightarrow h_f = \frac{4\tau_w}{\rho g}\frac{L}{d} \Rightarrow f\frac{L}{D}\frac{V^2}{2g} = \frac{4\tau_w}{\rho g}\frac{L}{d} \Rightarrow f = \frac{8\tau_w}{\rho V^2}$$

I: 对于层流 $f_{lam} = \dfrac{64}{Re_d}$

推导: $f_{lam} = \dfrac{8\tau_w}{\rho V^2} = \dfrac{8(8\mu\frac{V}{d})}{\rho V^2} = \dfrac{64}{\frac{\rho V d}{\mu}} = \dfrac{64}{Re_d}$

II: 对于湍流 $f_{tur} = \begin{cases} 0.316\, Re_d^{-\frac{1}{4}} & 4000<Re_d<10^5 \\ (1.8\log\frac{Re_d}{6.9})^{-2} & \text{Ref.9} \end{cases}$ (不考虑粗糙度时. f用这个试算).

推导: 圆管中湍流速度公式: $\dfrac{V}{u^*} = 2.44 \ln\dfrac{Ru^*}{\nu} + 1.34$

$\because u^* = (\frac{\tau_w}{\rho})^{\frac{1}{2}}$ $\therefore \frac{V}{u^*} = (\frac{\rho V^2}{\tau_w})^{\frac{1}{2}} = (\frac{8}{f})^{\frac{1}{2}}$ ①

$\frac{Ru^*}{\nu} = \frac{\frac{d}{2}V}{\nu}\frac{u^*}{V} = \frac{1}{2}Re_d(\frac{f}{8})^{\frac{1}{2}}$ ② 将①②代入湍流速度公式有 $\frac{1}{f^{\frac{1}{2}}} = 1.99\log(Re_d f^{\frac{1}{2}}) - 1.0$

$$\Rightarrow f = 0.316\, Re_d^{-\frac{1}{4}}$$

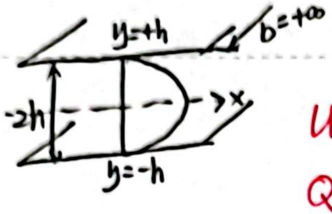
Non-Circular Ducts 非圆形管

定义: $D_h = \frac{4A}{P}$

A : cross-sectional area of actual flow 流体横截面积

P : wetted perimeter 润湿周长

模型一: flow between paralled plates



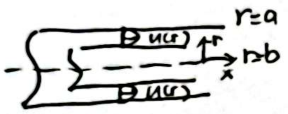
$$D_h = \frac{4A}{P} = \lim_{b\to\infty}\frac{4(2bh)}{2b+4h} = 4h$$

$u = u_{max}\left(1-\frac{y^2}{h^2}\right)$ $\quad u_{max} = \frac{h^2}{2\mu}\frac{\Delta P}{L}$ $\quad \tau_w = h\frac{\Delta P}{L} = \frac{3\mu V}{h}$

$Q = \frac{bh^3}{3\mu}\frac{\Delta P}{L}$ $\quad h_f = \frac{\Delta P}{\rho g} = \frac{3\mu L V}{\rho g h^2}$

$V = \frac{h^2}{3\mu}\frac{\Delta P}{L} = \frac{2}{3}u_{max}$

模型二. flow through a concentric annulus



$$D_h = \frac{4A}{P} = \frac{4\pi(a^2-b^2)}{2\pi(a+b)} = 2(a-b).$$

$f = h_f\frac{D_h}{L}\frac{2g}{V^2}$ $\quad V = \frac{Q}{\pi(a^2-b^2)}$

$f = \frac{64\zeta}{Re_{D_h}}$ $\quad \zeta = \frac{(a-b)^2(a^2-b^2)}{a^4-b^4-(a^2-b^2)^2/\ln\frac{a}{b}}$

$f = \frac{64}{Re_{eff}}$ $\quad Re_{eff} = \frac{1}{\zeta}Re_{D_h}$

管道中阀门等部件会产生阻力. 故: $h_{tot} = h_f + \sum h_m$

$$= \frac{V^2}{2g}\left(\frac{fL}{d} + \sum k\right)$$

查 P387 table 6.5

Fig 6.18~6.23表

Boundary layer flows 平板

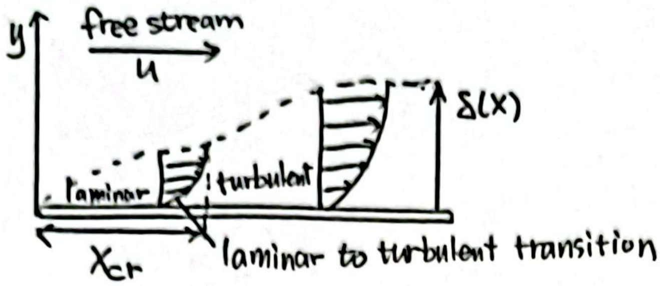
一. $Re_x = \frac{\rho U_\infty X}{\mu} = \frac{U_\infty X}{\nu}$

U_∞: characteristic flow velocity

X : characteristic flow dimension

$Re_{cr} = \frac{\rho U_\infty X_{cr}}{\mu} = 500000$

$\begin{cases} Re_x < 500000 & \text{laminar} \\ Re_x > 500000 & \text{turbulent} \end{cases}$



二. $\tau_w = \rho U^2 \frac{d\theta}{dx}$ (层流.湍流都可用)

推导(作业未布置): ex3.11 中用质量守恒+动量定律推出 $D(x) = \rho b U^2 \theta$ (b为宽度)

其中 momentum thickness $\theta = \int_0^\delta \frac{u}{U}(1-\frac{u}{U})dy$

又∵ $D(x) = b\int_0^x \tau_w(x)dx$ ∴ $\frac{dD}{dx} = b\tau_w = \rho b U^2 \frac{d\theta}{dx}$ ∴ $\tau_w = \rho U^2 \frac{d\theta}{dx}$

①. 层流中 $\frac{\delta}{x} = \frac{5}{Re_x^{\frac{1}{2}}}$

I: 近似法推 $\frac{\delta}{x} = \frac{5.5}{Re_x^{\frac{1}{2}}}$

推导: $u(x,y) = U(\frac{2y}{\delta} - \frac{y^2}{\delta^2})$

$$\left.\begin{aligned} \theta &= \int_0^\delta (\frac{2y}{\delta} - \frac{y^2}{\delta^2})(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2})dy = \frac{2}{15}\delta \\ \tau_w &= \mu \frac{\partial u}{\partial y}\Big|_{y=0} \approx \frac{2\mu U}{\delta} \end{aligned}\right\} \text{代入 } \tau_w = \rho U^2 \frac{d\theta}{dx} \Rightarrow \delta d\delta = 15\frac{\nu}{U}d$$

积分得 $\frac{1}{2}\delta^2 = \frac{15\nu x}{\cancel{Re_x}\,U} \Rightarrow \frac{\delta}{x} = 5.5(\frac{\nu}{Ux})^{\frac{1}{2}}$.

II. 理论解 $\frac{\delta}{x} = \frac{5}{Re_x^{\frac{1}{2}}}$

推导: 连续性方程: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

NS方程 (两维. 忽略重力):
$\rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$
$\rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$.

$v \ll u$ $\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$ $\frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}$ $Re_x = \frac{Ux}{\nu} \gg 1$

∴ NS方程中 $\underbrace{\rho(u\frac{\partial v}{\partial x})}_{\text{small}} + \underbrace{\rho(v\frac{\partial v}{\partial y})}_{\text{small}} = -\frac{\partial p}{\partial y} + \underbrace{\mu(\frac{\partial^2 v}{\partial x^2})}_{\text{very small}} + \underbrace{\mu(\frac{\partial^2 v}{\partial y^2})}_{\text{very small}}$

∴ $\frac{\partial p}{\partial y} \approx 0$ ∴ $p \approx p(x)$ only (y动量方程可忽略. 压力只沿边界层变化, 不穿过边界层)

⇒ NS方程的边界改进型: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx U\frac{dU}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$

其中 $\tau = \begin{cases} \mu\frac{\partial u}{\partial y} & \text{层流} \\ \mu\frac{\partial u}{\partial y} - \rho\overline{u'v'} & \text{湍流} \end{cases}$

后续解方程得 $\frac{\delta}{x} = \frac{5}{Re^{\frac{1}{2}}}$

② 湍流中 $\frac{\delta}{x} \approx \frac{0.16}{Re_x^{\frac{1}{7}}}$

推导：$\frac{u}{u^*} = \frac{1}{0.41}\ln\frac{\delta u^*}{\nu} + 5.$ $u^* = \left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}}$

定义式：$C_f = \frac{2\tau_w}{\rho U^2}$ $\frac{U}{u^*} = \left(\frac{\rho U^2}{\tau_w}\right)^{\frac{1}{2}} = \left(\frac{2}{C_f}\right)^{\frac{1}{2}}$ $\frac{\delta u^*}{\nu} = \frac{\delta}{\nu}\left(\frac{\tau_w}{\rho}\right)^{\frac{1}{2}} = \frac{\delta U}{\nu}\left(\frac{\tau_w}{\rho U^2}\right)^{\frac{1}{2}}$

$= Re_\delta\left(\frac{C_f}{2}\right)^{\frac{1}{2}}$

$\therefore \frac{u}{u^*} = \frac{1}{0.41}\ln\frac{\delta u^*}{\nu} + 5$ 可写成 $\left(\frac{2}{C_f}\right)^{\frac{1}{2}} = 2.44\ln\left[Re_\delta\left(\frac{C_f}{2}\right)^{\frac{1}{2}}\right] + 5$

实验得到 $C_f = 0.02\,Re_\delta^{-\frac{1}{6}}$

$$\left.\begin{aligned} \tau_w &= \rho U^2\frac{d\theta}{dx} \\ C_f &= \frac{2\tau_w}{\rho U^2} \end{aligned}\right\} \Rightarrow C_f = 2\frac{d\theta}{dx}$$

$\because$ 湍流速度分布可用 $\left(\frac{u}{U}\right)_{turb} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$ 近似

$\therefore \theta = \int_0^\delta \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\left[1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right]dy = \frac{7}{72}\delta$ $\therefore C_f = \frac{2d\theta}{dx} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right)$

又 $\because C_f = 0.02\,Re_\delta^{-\frac{1}{6}}$ $\therefore Re_\delta^{-\frac{1}{6}} = 9.72\frac{d\delta}{dx} = 9.72\frac{d(Re_\delta)}{d(Re_x)}$

$\therefore Re_\delta = 0.16\,Re_x^{\frac{6}{7}}$ $\therefore \frac{\delta}{x} = \frac{0.16}{Re_x^{\frac{1}{7}}}$

三、平板阻力系数 C_f

定义式：$C_f = \frac{2\tau_w}{\rho U^2}$ $C_D = \frac{2\frac{F_D}{A}}{\rho U^2}$

①光滑时：$C_f = \begin{cases} \frac{0.664}{Re_x^{\frac{1}{2}}} & \text{laminar} \\ \frac{0.027}{Re_x^{\frac{1}{7}}} & \text{turbulent} \end{cases}$

$C_D = \begin{cases} \frac{1.328}{Re_L^{\frac{1}{2}}} & \text{laminar} \\ \frac{0.031}{Re_L^{\frac{1}{7}}} & \text{turbulent} \end{cases}$

$F_{Drag} = C_D \frac{\rho U^2}{2} A$.

湍流的 C_f, C_D 可由 $\frac{L}{\varepsilon}$ 和 Re_L 查表知.

②粗糙时.

湍流 $\begin{cases} C_f = (2.87 + 1.58 \log \frac{x}{\varepsilon})^{-2.5} \\ C_D = (1.89 + 1.62 \log \frac{L}{\varepsilon})^{-2.5} \end{cases}$

$C_D = \begin{cases} \frac{0.031}{Re_L^{\frac{1}{7}}} - \frac{1440}{Re_L} & Re_{trans} = 5\times10^5 \\ \frac{0.031}{Re_L^{\frac{1}{7}}} - \frac{8700}{Re_L} & Re_{trans} = 3\times10^6 \end{cases}$

等容比热 $C_v = \frac{R}{k-1}$

等压比热 $C_p = \frac{kR}{k-1}$ $h = C_p T$

理想气体状态方程 $\rho = \frac{p}{RT}$

滞止焓 $h_0 = h + \frac{1}{2}V^2 = const$

绝热，
$$\begin{cases} \frac{T_0}{T} = 1 + \frac{k-1}{2} Ma^2 \\ \frac{a_0}{a} = \left(\frac{T_0}{T}\right)^{\frac{1}{2}} = \left(1 + \frac{k-1}{2} Ma^2\right)^{\frac{1}{2}} \end{cases}$$

adiabatic

等熵
$$\begin{cases} \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left[1 + \frac{1}{2}(k-1) Ma^2\right]^{\frac{k}{k-1}} \\ \frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left[1 + \frac{k-1}{2} Ma^2\right]^{\frac{1}{k-1}} \end{cases}$$

isentropic

$T_0 = T_1 + \frac{V^2}{2C_p}$.